

Global well-posedness for Euler-Nernst-Planck-Possion system in dimension two

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Abstract

In this paper, we study the Cauchy problem of the Euler-Nernst-Planck-Possion system. We obtain global well-posedness for the system in dimension $d = 2$ for any initial data in $H^{s_1}(\mathbb{R}^2) \times H^{s_2}(\mathbb{R}^2) \times H^{s_2}(\mathbb{R}^2)$ under certain conditions of s_1 and s_2 .

2010 Mathematics Subject Classification: 35Q35, 35K15, 76N10.

Keywords: Electrohydrodynamics; Euler-Nernst-Planck-Possion system; Global well-posedness; Littlewood-Paley theory.

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1 Introduction

In this paper, we study the Cauchy problem of the following nonlinear system:

$$(01) \quad \begin{cases} u_t + u \cdot \nabla u - \nu \Delta u + \nabla P = \Delta \phi \nabla \phi, & t > 0, x \in \mathbb{R}^d, \\ \nabla \cdot u = 0, & t > 0, x \in \mathbb{R}^d, \\ n_t + u \cdot \nabla n = \nabla \cdot (\nabla n - n \nabla \phi), & t > 0, x \in \mathbb{R}^d, \\ p_t + u \cdot \nabla p = \nabla \cdot (\nabla p + p \nabla \phi), & t > 0, x \in \mathbb{R}^d, \\ \Delta \phi = n - p, & t > 0, x \in \mathbb{R}^d, \\ (u, n, p)|_{t=0} = (u_0, n_0, p_0), & x \in \mathbb{R}^d. \end{cases}$$

Here $u(t, x)$ is a vector in \mathbb{R}^d , $P(t, x)$, $n(t, x)$, $p(t, x)$ and $\phi(t, x)$ are scalars. The first two equations of the system (01) are the conservation equations of the incompressible flow. u denotes the velocity field, P denotes the pressure, $\nu \geq 0$ denotes the fluid viscosity and ϕ denotes the electrostatic potential caused by the net charged particles. The third and the fourth equations of the system (01), which are the Nernst-Planck equations modified by the convective terms $u \cdot \nabla n$ and $u \cdot \nabla p$, model the balance between diffusion and convective transport of charge densities by flow and electric fields. n and p are the densities of the negative and positive charged particles. They are coupled by the Poisson equation (the fifth equation). The system (01) arises from electrohydrodynamics, which describing the dynamic coupling between incompressible flows and diffuse charge systems finds application in biology, chemistry and pharmacology. See [2, 5, 8, 9] for more details.

If the fluid viscosity $\nu > 0$, The above system (01) is the so called Navier-Stokes-Nernst-Planck-Possion (*NSNPP*) system, and it has been studied by several authors. Schmuck [11] and Ryham [10] obtained the global existence of weak solutions in a bounded domain Ω in dimension $d \leq 3$ with Neumann and Dirichlet boundary conditions respectively. By using elaborate energy analysis, Li [7] studied the quasineutral limit in periodic domain. When $\Omega = \mathbb{R}^n$, Joseph [5] established the existence of a unique smooth local solution for smooth initial data by making use of Kato's semigroup ideas. The author also established the stability under the inviscid limit $\nu \rightarrow 0$. Zhao et al. [3, 4, 13, 14] studied the local and global well-posedness in the critical Lebesgue spaces, modulation spaces, Triebel-Lizorkin spaces and Besov spaces by using the Banach fixed point theorem.

If, on the other hand, $\nu = 0$, the above system (01) is the Euler-Nernst-Planck-Possion (*ENPP*) system. Recently, Zhang and Yin [12] proved the local well-posedness for the *ENPP* system in Besov spaces in dimension $d \geq 2$.

The purpose of this paper is to get the global existence for the *ENPP* system in dimension $d = 2$. Motivated by [1] for the study of the Euler system, we first introduce the following modified system

$$(02) \quad \begin{cases} u_t + u \cdot \nabla u + \Pi(u, u) = \mathcal{P}((\nabla \cdot \xi)\xi), \\ n_t + \nabla \cdot (un) - \Delta n = -\nabla \cdot (n\xi), \\ p_t + \nabla \cdot (up) - \Delta p = \nabla \cdot (p\xi), \\ \xi = -\nabla(-\Delta)^{-1}(n - p), \end{cases}$$

where \mathcal{P} is the Leray projector defined as $\mathcal{P} = Id + \nabla(-\Delta)^{-1}\nabla \cdot$, and $\Pi(\cdot, \cdot)$ is a bilinear operator defined by $\Pi(u, v) = \sum_{j=1}^5 \Pi_j(u, v)$, with

$$\begin{aligned} \Pi_1(u, v) &= \nabla |D|^{-2} T_{\partial_i u^j} \partial_j v^i, & \Pi_2(u, v) &= \nabla |D|^{-2} T_{\partial_j v^i} \partial_i u^j, \\ \Pi_3(u, v) &= \nabla |D|^{-2} \partial_i \partial_j (I - \Delta_{-1}) R(u^i, v^j), & \Pi_4(u, v) &= \theta E_d * \nabla \partial_i \partial_j \Delta_{-1} R(u^i, v^j), \\ \Pi_5(u, v) &= \nabla \partial_i \partial_j ((1 - \theta) E_d) * \Delta_{-1} R(u^i, v^j). \end{aligned}$$

Here θ is a function of $\mathcal{D}(B(0, 2))$ with value 1 on $B(0, 1)$, E_d stands for the fundamental solution of $-\Delta$, and $|D|^{-2}$ denotes the Fourier multiplier with symbol $|\xi|^{-2}$. See Section 2 for the definitions of T and R .

We deduce from the second to the fourth equations of the system (02) that the dynamic equations of $(n + p, \xi)$ are

$$\begin{cases} (n + p)_t + \nabla \cdot (u(n + p)) - \Delta(n + p) = -\nabla \cdot ((\nabla \cdot \xi)\xi), \\ \xi_t - \Delta \xi + (-\nabla(-\Delta)^{-1}\nabla \cdot)(u(\nabla \cdot \xi)) = -(-\nabla(-\Delta)^{-1}\nabla \cdot)((n + p)\xi). \end{cases}$$

Denote $\mathcal{L} = -\nabla(-\Delta)^{-1}\nabla \cdot = Id - \mathcal{P}$. We then introduce the following system

$$(03) \quad \begin{cases} u_t + u \cdot \nabla u + \Pi(u, u) = \mathcal{P}((\nabla \cdot \xi)\xi), \\ z_t + \nabla \cdot (uz) - \Delta z = -\nabla \cdot ((\nabla \cdot \xi)\xi), \\ \xi_t - \Delta \xi + \mathcal{L}(u(\nabla \cdot \xi)) = -\mathcal{L}(z\xi). \end{cases}$$

Note that, by means of basic energy argument, the terms $\langle \mathcal{P}((\nabla \cdot \xi)\xi), u \rangle$ and $\langle \mathcal{L}(u(\nabla \cdot \xi)), \xi \rangle$ can be canceled out, which plays an important role in the proof of global existence.

We point out that in [12], the term $\nabla \phi = \nabla(-\Delta)^{-1}(p-n)$ was controlled by $n-p$ through the Hardy-Littlewood-Sobolev inequality, i.e., $\|\nabla \phi\|_{L^q} \lesssim \|n-p\|_{L^p}$ with $1 < p < d$. Whereas, in this paper $n-p = \nabla \cdot \xi$ is controlled by ξ through $\|n-p\|_{H^s} \lesssim \|\xi\|_{H^{s+1}}$. Hence, these two papers solve the *ENPP* system in different function spaces.

We can now state our main results:

Theorem 1.1. *Let $d \geq 2$, $(s_1, s_2) \in \mathbb{R}^2$, satisfying*

$$(1.1) \quad s_1 > 1 + \frac{d}{2}, \text{ and } s_2 + \frac{3}{2} > s_1 \geq s_2 + 1.$$

There exists constants c and r , depending only on s_1, s_2 and d , such that for $(u_0, z_0, \xi_0) \in H^{s_1}(\mathbb{R}^d) \times H^{s_2}(\mathbb{R}^d) \times H^{s_2+1}(\mathbb{R}^d)$, with $\nabla \cdot u_0 = 0$, $\xi_0 = -\nabla(-\Delta)^{-1}a_0$ for some $a_0 \in H^{s_2}(\mathbb{R}^d)$, and $z_0 \pm \nabla \cdot \xi_0 \geq 0$, there exists a time

$$T \geq \frac{c}{1 + (\|u_0\|_{H^{s_1}(\mathbb{R}^d)} + \|z_0\|_{H^{s_2}(\mathbb{R}^d)} + \|\xi_0\|_{H^{s_2+1}(\mathbb{R}^d)})^r},$$

such that the system (03) has a unique solution (u, z, ξ) on $[0, T] \times \mathbb{R}^d$ satisfying

$$(u, z, \xi) \in \tilde{L}_T^\infty(H^{s_1}(\mathbb{R}^d)) \times (\tilde{L}_T^\infty(H^{s_2}(\mathbb{R}^d)) \cap \tilde{L}_T^1(H^{s_2+2}(\mathbb{R}^d))) \times (\tilde{L}_T^\infty(H^{s_2+1}(\mathbb{R}^d)) \cap \tilde{L}_T^1(H^{s_2+3}(\mathbb{R}^d))),$$

and (u, z, ξ) is continuous in time with values in $H^{s_1} \times H^{s_2} \times H^{s_2+1}$.

Moreover, $\nabla \cdot u = 0$, $z \pm \nabla \cdot \xi \geq 0$, a.e. on $[0, T] \times \mathbb{R}^d$, and $\mathcal{L}\xi = \xi$.

Finally, if $d = 2$ and $s_2 > 1$, then the solution (u, z, ξ) is global.

Remark 1.2. *We mention that the restriction $s_1 > 1 + \frac{d}{2}$ is due to the same reasons as illustrated for the Euler equation in [1]. $s_2 + \frac{3}{2} > s_1 \geq s_2 + 1$ is caused by the coupling between u and ξ . In fact, owing to the properties of the transport flow, (u, ξ) is expected to be in $\tilde{L}_T^\infty(H^{s_1}(\mathbb{R}^d)) \times \tilde{L}_T^r(H^{s_2+1+\frac{2}{r}}(\mathbb{R}^d))$ with $r \in [1, \infty]$. Due to the product laws in Besov spaces, ab is less regular than a or b . Thus in order to control the term $(\nabla \cdot \xi)\xi$ in the first equation of the system, we have to assume $s_2 + 1 + \frac{2}{r_1} \geq (>)s_1$, $s_2 + \frac{2}{r_1} > (\geq)s_1$, for some $r_1 \in [1, \infty]$ and $\frac{1}{r_1} + \frac{1}{r_1} = 1$, which implies that $s_2 + \frac{3}{2} > s_1$. Similar reason for the term $u(\nabla \cdot \xi)$ requires $s_1 \geq s_2 + 1$.*

Theorem 1.3. *Let $d = 2$, $(s_1, s_2) \in \mathbb{R}^2$,*

$$(1.2) \quad s_1 > 2, \quad s_2 > 1, \text{ and } s_2 + \frac{3}{2} > s_1 \geq s_2 + 1.$$

*Then for any $(u_0, n_0, p_0) \in H^{s_1}(\mathbb{R}^2) \times H^{s_2}(\mathbb{R}^2) \times H^{s_2}(\mathbb{R}^2)$, with $\nabla \cdot u_0 = 0$, $\nabla(-\Delta)^{-1}(n_0 - p_0) \in H^{s_2+1}(\mathbb{R}^2)$, and $n_0, p_0 \geq 0$, the *ENPP* system has a solution (u, n, p, P, ϕ) on $\mathbb{R}^+ \times \mathbb{R}^2$ satisfying*

$$\begin{aligned} (u, n, p) &\in \tilde{L}^\infty(\mathbb{R}^+; H^{s_1}(\mathbb{R}^2)) \times \left(\tilde{L}^\infty(\mathbb{R}^+; H^{s_2}(\mathbb{R}^2)) \cap \tilde{L}^1(\mathbb{R}^+; H^{s_2+2}(\mathbb{R}^2)) \right)^2, \\ -\nabla(-\Delta)^{-1}(n-p) &\in \tilde{L}^\infty(\mathbb{R}^+; H^{s_2+1}(\mathbb{R}^2)) \cap \tilde{L}^1(\mathbb{R}^+; H^{s_2+3}(\mathbb{R}^2)), \\ P, \Phi &\in L^\infty(\mathbb{R}^+; BMO(\mathbb{R}^2)). \end{aligned}$$

Moreover, if $(\tilde{u}, \tilde{n}, \tilde{p}, \tilde{P}, \tilde{\phi})$ also satisfies the ENPP system with the same initial data and belongs to the above class, then $(u, n, p) = (\tilde{u}, \tilde{n}, \tilde{p})$, and $(\nabla P, \nabla \phi) = (\nabla \tilde{P}, \nabla \tilde{\phi})$.

Finally, (u, n, p) is continuous in time with values in $H^{s_1} \times H^{s_2} \times H^{s_2}$, and $n, p \geq 0$, a.e. on $\mathbb{R}^+ \times \mathbb{R}^2$.

Remark 1.4. We mention that under an improved condition 1.2, Theorem 1.3 may hold true for the NSNPP system. We will present this result in another paper.

Throughout the paper, $C > 0$ stands for a generic constant and $c > 0$ a small constant. We shall sometimes use the notation $A \lesssim B$ to denote the relation $A \leq CB$. For simplicity, we write L^p , H^s and $B_{p,r}^s$ for the spaces $L^p(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, and $B_{p,r}^s(\mathbb{R}^d)$, respectively.

The remain part of this paper is organized as follows. In Section 2, we recall some basic facts about Littlewood-Paley theory and Besov spaces. Section 3 is devoted to the proof of Theorem 1.1. Finally, we give a proof of Theorem 1.3 by using Theorem 1.1.

2 Preliminaries

2.1. The nonhomogeneous Besov spaces

We first define the Littlewood-Paley decomposition.

Lemma 2.1. [1] Let $\mathcal{C} = \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ be an annulus. There exist radial functions χ and φ valued in the interval $[0, 1]$, belonging respectively to $\mathcal{D}(B(0, \frac{4}{3}))$ and $\mathcal{D}(\mathcal{C})$, such that

$$\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1.$$

The nonhomogeneous dyadic blocks Δ_j and the nonhomogeneous low-frequency cut-off operator S_j are then defined as follows:

$$\begin{aligned} \Delta_j u &= 0 \quad \text{if } j \leq -2, & \Delta_{-1} u &= \chi(D)u, \\ \Delta_j u &= \varphi(2^{-j}D)u \quad \text{if } j \geq 0, & S_j u &= \sum_{j' \leq j-1} \Delta_{j'} u, \quad \text{for } j \in \mathbb{Z}. \end{aligned}$$

We may now introduce the nonhomogeneous Besov spaces.

Definition 2.2. Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B_{p,r}^s$ consists of all tempered distributions u such that

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left\| (2^{js} \|\Delta_j u\|_{L^p})_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})} < \infty.$$

The Sobolev space can be defined as follows:

Definition 2.3. For $s \in \mathbb{R}$,

$$H^s = \{u \in \mathcal{S}' ; \|u\|_{H^s} = \left(\sum_{j=-1}^{\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty\}.$$

Remark 2.4. For any $s \in \mathbb{R}$, the Besov space $B_{2,2}^s$ coincides with the Sobolev space H^s .

Lemma 2.5. The set $B_{p,r}^s$ is a Banach space, and satisfies the Fatou property, namely, if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B_{p,r}^s$, then an element u of $B_{p,r}^s$ and a subsequence $u_{\psi(n)}$ exist such that

$$\lim_{n \rightarrow \infty} u_{\psi(n)} = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{B_{p,r}^s} \leq C \liminf_{n \rightarrow \infty} \|u_{\psi(n)}\|_{B_{p,r}^s}.$$

In addition to the general time-space $L_T^\rho(B_{p,r}^s)$, we introduce the following mixed time-space $\tilde{L}_T^\rho(B_{p,r}^s)$.

Definition 2.6. For all $T > 0$, $s \in \mathbb{R}$, and $1 \leq r, \rho \leq \infty$, we define the space $\tilde{L}_T^\rho(B_{p,r}^s)$ the set of tempered distributions u over $(0, T) \times \mathbb{R}^d$, such that

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \stackrel{\text{def}}{=} \|2^{js} \|\Delta_j u\|_{L_T^\rho(L^p)}\|_{l^r(\mathbb{Z})} < \infty.$$

It follows from the Minkowski inequality that

$$\|u\|_{L_T^\rho(B_{p,r}^s)} \leq \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \text{ if } r \leq \rho, \quad \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq \|u\|_{L_T^\rho(B_{p,r}^s)} \text{ if } r \geq \rho.$$

Let's then recall Bernstein-Type lemmas.

Lemma 2.7. [1] (Bernstein inequalities) Let \mathcal{C} be an annulus and \mathcal{B} a ball. A constant C exists such that for any nonnegative integer k , any couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function u of L^p , we have

$$\begin{aligned} \text{Supp } \hat{u} \subset \lambda \mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \end{aligned}$$

We state the following embedding and interpolation inequalities.

Lemma 2.8. [1] Let $1 \leq p_1 \leq p_2 \leq \infty$ and $\infty \leq r_1 \leq r_2 \leq \infty$. Then for any real number s , we have

$$B_{p_1, r_1}^s \hookrightarrow B_{p_2, r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}.$$

Lemma 2.9. [1] If s_1 and s_2 are real numbers such that $s_1 < s_2$, $\theta \in (0, 1)$ and $1 \leq p, r \leq \infty$, then we have

$$B_{p, \infty}^{s_2} \hookrightarrow B_{p, 1}^{s_1}, \quad \text{and} \quad \|u\|_{B_{p, r}^{\theta s_1 + (1-\theta)s_2}} \leq \|u\|_{B_{p, r}^{s_1}}^\theta \|u\|_{B_{p, r}^{s_2}}^{1-\theta}.$$

In the sequel, we will frequently use the Bony decomposition:

$$uv = T_v u + T_u v + R(u, v),$$

with

$$\begin{aligned} R(u, v) &= \sum_{|k-j| \leq 1} \Delta_k u \Delta_j v, \\ T_u v &= \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v = \sum_{j \geq 1} S_{j-1} u \Delta_j ((Id - \Delta_{-1})v), \end{aligned}$$

where operator T is called “paraproduct”, whereas R is called “remainder”.

Lemma 2.10. *A constant C exists which satisfies the following inequalities for any couple of real numbers (s, t) with t negative and any $(p, p_1, p_2, r, r_1, r_2)$ in $[1, \infty]^6$:*

$$\begin{aligned} \|T\|_{\mathcal{L}(L^{p_1} \times B_{p_2, r}^s; B_{p, r}^s)} &\leq C^{|s|+1}, \\ \|T\|_{\mathcal{L}(B_{p_1, r_1}^t \times B_{p_2, r_2}^s; B_{p, r}^{s+t})} &\leq \frac{C^{|s+t|+1}}{-t}, \end{aligned}$$

with $\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, $\frac{1}{r} \stackrel{\text{def}}{=} \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$.

Proof. The proof of this lemma can be easily deduced from substituting the estimate

$$\|S_{j-1}u \triangle_j v\|_{L^p} \leq \|S_{j-1}u\|_{L^{p_1}} \|\triangle_j v\|_{L^{p_2}},$$

for the estimate

$$\|S_{j-1}u \triangle_j v\|_{L^p} \leq \|S_{j-1}u\|_{L^\infty} \|\triangle_j v\|_{L^p}$$

in the proof of Theorem 2.82 in [1]. It is thus omitted. \square

Lemma 2.11. *[1] A constant C exists which satisfies the following inequalities. Let (s_1, s_2) be in \mathbb{R}^2 and (p_1, p_2, r_1, r_2) be in $[1, \infty]^4$. Assume that*

$$\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

If $s_1 + s_2 > 0$, then we have, for any (u, v) in $B_{p_1, r_1}^{s_1} \times B_{p_2, r_2}^{s_2}$,

$$\|R(u, v)\|_{B_{p, r}^{s_1+s_2}} \leq \frac{C^{|s_1+s_2|+1}}{s_1 + s_2} \|u\|_{B_{p_1, r_1}^{s_1}} \|v\|_{B_{p_2, r_2}^{s_2}}.$$

If $r = 1$ and $s_1 + s_2 = 0$, then we have, for any (u, v) in $B_{p_1, r_1}^{s_1} \times B_{p_2, r_2}^{s_2}$,

$$\|R(u, v)\|_{B_{p, \infty}^0} \leq C \|u\|_{B_{p_1, r_1}^{s_1}} \|v\|_{B_{p_2, r_2}^{s_2}}.$$

Lemma 2.12. *Let $s + \frac{1}{2} > \frac{d}{2}$. A constant C exists such that*

$$\begin{aligned} \|uv\|_{H^s} &\lesssim \|u\|_{H^{s+\frac{1}{2}}} \|v\|_{H^s}, \\ \|uv\|_{H^{s+1}} &\lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}}. \end{aligned}$$

Proof. By using Bony's decomposition combined with Lemmas 2.10-2.11, we have

$$\begin{aligned} \|uv\|_{H^s} &\lesssim \|T_u v\|_{H^s} + \|R(u, v)\|_{H^s} + \|T_v u\|_{H^s} \\ &\lesssim \|u\|_{L^\infty} \|v\|_{H^s} + \|u\|_{B_{\infty, \infty}^0} \|v\|_{H^s} + \|v\|_{B_{\infty, \infty}^{-\frac{1}{2}}} \|u\|_{H^{s+\frac{1}{2}}} \\ &\lesssim \|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{H^{\frac{d}{2}-\frac{1}{2}}} \|u\|_{H^{s+\frac{1}{2}}} \\ &\lesssim \|u\|_{H^{s+\frac{1}{2}}} \|v\|_{H^s}, \end{aligned}$$

where we have used $s > \frac{d}{2} - \frac{1}{2} > 0$, and $H^{s+\frac{1}{2}} \hookrightarrow L^\infty$. Similarly,

$$\|uv\|_{H^{s+1}} \lesssim \|u\|_{L^\infty} \|v\|_{H^{s+1}} + \|v\|_{L^\infty} \|u\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}},$$

where we have used $H^{s+1} \hookrightarrow L^\infty$. We thus obtain the desired inequalities. \square

We mention that all the properties of continuity for the paraproduct and remainder remain true in the mixed time-space $\tilde{L}_T^\rho(B_{p,r}^s)$.

Finally, we state the following commutator estimates.

Lemma 2.13. [1] *Let v be a vector field over \mathbb{R}^d , define $R_j = [v \cdot \nabla, \triangle_j]f$. Let $\sigma > 0$ (or $\sigma > -1$, if $\nabla \cdot v = 0$), $1 \leq r \leq \infty$, $1 \leq p \leq p_1 \leq \infty$, and $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$. Then*

$$\left\| 2^{j\sigma} \|R_j\|_{L^p} \right\|_{l^r} \leq C \left(\|\nabla v\|_{L^\infty} \|f\|_{B_{p,r}^\sigma} + \|\nabla f\|_{L^{p_2}} \|\nabla v\|_{B_{p_1,r}^{\sigma-1}} \right).$$

2.2. A priori estimates for transport and transport-diffusion equations

Let us state some classical a priori estimates for transport equations and transport-diffusion equations.

Lemma 2.14. [1] *Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$. Assume that*

$$(2.1) \quad s \geq -d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{or} \quad s \geq -1 - d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{if } \nabla \cdot v = 0$$

with strict inequality if $r < \infty$.

There exists a constant C , depending only on d, p, p_1, r and s , such that for all solutions $f \in L^\infty([0, T]; B_{p,r}^s)$ of the transport equation

$$(2.2) \quad \begin{cases} \partial_t f + v \cdot \nabla f = g \\ f|_{t=0} = f_0, \end{cases}$$

with initial data f_0 in $B_{p,r}^s$, and g in $L^1([0, T]; B_{p,r}^s)$, we have, for a.e. $t \in [0, T]$,

$$(2.3) \quad \|f\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \left(\|f_0\|_{B_{p,r}^s} + \int_0^t \exp(-CV_{p_1}(t')) \|g(t')\|_{B_{p,r}^s} dt' \right) \exp(CV_{p_1}(t)),$$

with, if the inequality is strict in (2.1),

$$(2.4) \quad V_{p_1}'(t) = \begin{cases} \|\nabla v(t)\|_{B_{p_1,r}^{s-1}}, & \text{if } s > 1 + \frac{d}{p_1} \text{ or } s = 1 + \frac{d}{p_1}, \quad r = 1, \\ \|\nabla v(t)\|_{B_{p_1,\infty}^{\frac{d}{p_1}} \cap L^\infty}, & \text{if } s < 1 + \frac{d}{p_1} \end{cases}$$

and, if equality holds in (2.1) and $r = \infty$,

$$V_{p_1}' = \|\nabla v(t)\|_{B_{p_1,1}^{\frac{d}{p_1}}}.$$

If $f = v$, then for all $s > 0$ ($s > -1$, if $\nabla \cdot u = 0$), the estimate (2.3) holds with

$$V_{p_1}'(t) = \|\nabla u\|_{L^\infty}.$$

Lemma 2.15. [1] *Let $1 \leq p_1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $s \in \mathbb{R}$ satisfy (2.10), and let V_{p_1} be defined as in Lemma 2.14.*

There exists a constant C which depends only on d, r, s and $s - 1 - \frac{d}{p_1}$ and is such that for any smooth solution f of the transport diffusion equation

$$(2.5) \quad \begin{cases} \partial_t f + v \cdot \nabla f - \nu \triangle f = g \\ f|_{t=0} = f_0, \end{cases}$$

we have

$$\begin{aligned} \nu^{\frac{1}{p}} \|f\|_{\tilde{L}_T^p(B_{p,r}^{s+\frac{2}{p}})} &\leq C e^{C(1+\nu T)^{\frac{1}{p}} V_{p_1}(T)} \left((1+\nu T)^{\frac{1}{p}} \|f_0\|_{B_{p,r}^s} \right. \\ &\quad \left. + (1+\nu T)^{1+\frac{1}{p}-\frac{1}{p_1}} \nu^{\frac{1}{p_1}-1} \|g\|_{\tilde{L}_T^{\rho_1}(B_{p,r}^{s-2+\frac{2}{\rho_1}})} \right), \end{aligned}$$

where $1 \leq \rho_1 \leq \rho \leq \infty$.

2.3. The operator $\Pi(\cdot, \cdot)$

We recall some basic results for $\Pi(\cdot, \cdot)$. See [1] (Pages 296-300) for further details.

Lemma 2.16. [1] For all $s > -1$, and $1 \leq p, r \leq \infty$, there exists a constant C such that

$$\|\Pi(v, w)\|_{B_{p,r}^s} \leq C(\|v\|_{C^{0,1}} \|w\|_{B_{p,r}^s} + \|w\|_{C^{0,1}} \|v\|_{B_{p,r}^s}).$$

Moreover, there exists a bilinear operator P_Π such that $\Pi(v, w) = \nabla P_\Pi(v, w)$, and

$$\|P_\Pi(v, w)\|_{B_{p,r}^{s+1}} \leq C(\|v\|_{C^{0,1}} \|w\|_{B_{p,r}^s} + \|w\|_{C^{0,1}} \|v\|_{B_{p,r}^s}), \text{ if } 1 < p < \infty.$$

Lemma 2.17. [1] For all $-1 < s < \frac{d}{p} + 1$, and $1 \leq p, r \leq \infty$, we have

$$\|\Pi(v, w)\|_{B_{p,r}^s} \leq C \left(\|v\|_{C^{0,1}} \|w\|_{B_{p,r}^s} + \|w\|_{B_{\infty,\infty}^{s-\frac{d}{p}}} \|\nabla v\|_{B_{p,r}^{\frac{d}{p}}} \right).$$

Lemma 2.18. [1] For all $s > 1$, and $1 \leq p, r \leq \infty$, there exists a constant C such that

$$\|\nabla \cdot \Pi(v, w) + \text{tr}(Dv, Dw)\|_{B_{p,r}^{s-1}} \leq C \left(\|\nabla \cdot v\|_{B_{\infty,\infty}^0} \|w\|_{B_{p,r}^s} + \|\nabla \cdot w\|_{B_{\infty,\infty}^0} \|v\|_{B_{p,r}^s} \right).$$

2.4. The space $B_{\infty,\infty}^1$

The space $B_{\infty,\infty}^1$ plays an important role in dealing with the global existence. In this section, we introduce an interpolation inequality involving $B_{\infty,\infty}^1$.

Definition 2.19. Let α be in $(0, 1]$. A modulus of continuity is any nondecreasing nonzero continuous function $\mu : [0, \alpha] \rightarrow \mathbb{R}^+$ such that $\mu(0) = 0$. The modulus of continuity μ is admissible if, in addition, the function Γ defined for $y \geq \frac{1}{\alpha}$ by

$$\Gamma(y) \stackrel{\text{def}}{=} y\mu\left(\frac{1}{y}\right)$$

is nondecreasing and satisfies, for some constant C and all $x \geq \frac{1}{\alpha}$,

$$\int_x^\infty \frac{1}{y^2} \Gamma(y) dy \leq C \frac{\Gamma(x)}{x}.$$

Definition 2.20. Let μ be a modulus of continuity and (X, d) a metric space. We denote by $C_\mu(X)$ the set of bounded, continuous, real-valued functions u over X such that

$$\|u\|_{C_\mu} \stackrel{\text{def}}{=} \|u\|_{L^\infty} + \sup_{0 < d(x,y) \leq \alpha} \frac{|u(x) - u(y)|}{\mu(d(x,y))} < \infty.$$

Remark 2.21. [1] Let $\alpha = 1$. The function $\mu(r) = r(1 - \log r)$ is an admissible modulus of continuity, and the space C_μ contains $B_{\infty,\infty}^1$, more precisely, $B_{\infty,\infty}^1 \hookrightarrow C_\mu$.

Lemma 2.22. [1] Let μ be an admissible modulus of continuity. There exists a constant C such that for any $\varepsilon \in (0, 1]$, u in $C^{1,\varepsilon}$, and positive Λ , we have

$$\|\nabla u\|_{L^\infty} \leq C \left(\frac{\|u\|_{C_\mu} + \Lambda}{\varepsilon} + \|u\|_{C_\mu} \Gamma \left(\left(\frac{\|\nabla u\|_{C^{0,\varepsilon}}}{\|u\|_{C_\mu} + \Lambda} \right)^{\frac{1}{\varepsilon}} \right) \right)$$

whenever $\|u\|_{C_\mu} + \Lambda \leq (\frac{\alpha}{2})^\varepsilon \|\nabla u\|_{C^{0,\varepsilon}}$.

3 Proof of Theorem 1.1

To begin, we denote $\varepsilon = s_2 + \frac{3}{2} - s_1$, and $\varepsilon_0 = \min(\frac{1}{2}, \varepsilon)$. We mention that the condition (1.1) implies that

$$(3.1) \quad s_2 + \frac{1}{2} > s_1 - 1 > \frac{d}{2},$$

which will be frequently used.

3.1. Existence for the system (03)

3.1.1 First step: Construction of approximate solutions and uniform bounds

In order to define a sequence $(u^m, z^m, \xi^m)|_{m \in \mathbb{N}}$ of global approximate solutions to the system (03), we use an iterative scheme. First we set $u^0 = u_0$, $z^0 = e^{t\Delta} z_0$, $\xi^0 = e^{t\Delta} \xi_0$. Thanks to Lemma 2.15, it is easy to see that

$$(u^0, z^0, \xi^0) \in \tilde{L}_{loc}^\infty(\mathbb{R}^+; H^{s_1}) \times \left(\tilde{L}_{loc}^\infty(\mathbb{R}^+; H^{s_2}) \cap \tilde{L}_{loc}^1(\mathbb{R}^+; H^{s_2+1}) \right)^2,$$

and

$$\begin{aligned} & \|u^0\|_{\tilde{L}_t^\infty(H^{s_1})} + \|z^0\|_{\tilde{L}_t^\infty(H^{s_2}) \cap \tilde{L}_t^1(H^{s_2+2})} + \|\xi^0\|_{\tilde{L}_t^\infty(H^{s_2+1}) \cap \tilde{L}_t^1(H^{s_2+3})} \\ & \leq C(1+t)(\|u_0\|_{H^{s_1}} + \|z_0\|_{H^{s_2}} + \|\xi_0\|_{H^{s_2+1}}). \end{aligned}$$

Then, assuming that

$$(u^m, z^m, \xi^m) \in \tilde{L}_{loc}^\infty(\mathbb{R}^+; H^{s_1}) \times \left(\tilde{L}_{loc}^\infty(\mathbb{R}^+; H^{s_2}) \cap \tilde{L}_{loc}^1(\mathbb{R}^+; H^{s_2+2}) \right) \times \left(\tilde{L}_{loc}^\infty(\mathbb{R}^+; H^{s_2+1}) \cap \tilde{L}_{loc}^1(\mathbb{R}^+; H^{s_2+3}) \right),$$

we solve the following linear system:

$$(3.2) \quad \begin{cases} u_t^{m+1} + u^m \cdot \nabla u^{m+1} = -\Pi(u^m, u^m) - \mathcal{P}((\nabla \cdot \xi^m) \xi^m), \\ z_t^{m+1} - \Delta z^{m+1} = -\nabla \cdot (u^m z^m) - \nabla \cdot ((\nabla \cdot \xi^m) \xi^m), \\ \xi_t^{m+1} - \Delta \xi^{m+1} = -\mathcal{L}(u^m (\nabla \cdot \xi^m)) - \mathcal{L}(z^m \xi^m), \\ (u^{m+1}, z^{m+1}, \xi^{m+1})|_{t=0} = (u_0, z_0, \xi_0). \end{cases}$$

Lemma 2.14 ensures that

$$(3.3) \quad \|u^{m+1}\|_{\tilde{L}_t^\infty(H^{s_1})} \lesssim \exp(C \int_0^t \|u^m\|_{H^{s_1}} dt') \left(\|u_0\|_{H^{s_1}} + \|\Pi(u^m, u^m)\|_{\tilde{L}_t^1(H^{s_1})} + \|\mathcal{P}((\nabla \cdot \xi^m)\xi^m)\|_{\tilde{L}_t^1(H^{s_1})} \right).$$

Using Lemma 2.16, we get

$$(3.4) \quad \|\Pi(u, u)\|_{\tilde{L}_t^1(H^{s_1})} \lesssim \|u\|_{\tilde{L}_t^\infty(H^{s_1})} \|u\|_{\tilde{L}_t^\infty(H^{s_1})} t,$$

where we have used the fact that $H^{s_1} \hookrightarrow C^{0,1}$.

As for the term $\mathcal{P}((\nabla \cdot \xi^m)\xi^m)$, by taking advantage of Bony's decomposition and of Lemmas 2.10-2.11, we have

$$(3.5) \quad \begin{aligned} \|\mathcal{P}((\nabla \cdot \xi^m)\xi^m)\|_{H^{s_1}} &\lesssim \|(\nabla \cdot \xi^m)\xi^m\|_{H^{s_1}} \\ &\lesssim \|\nabla \cdot \xi^m\|_{B_{\infty,\infty}^{s_1-(s_2+\frac{3}{2})}} \|\xi\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m\|_{B_{\infty,\infty}^{s_1-(s_2+\frac{3}{2})}} \|\nabla \cdot \xi\|_{H^{s_2+\frac{3}{2}}} \\ &\lesssim \|\nabla \cdot \xi^m\|_{H^{s_1-(s_2+\frac{3}{2})+\frac{d}{2}}} \|\xi\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m\|_{H^{s_1-(s_2+\frac{3}{2})+\frac{d}{2}}} \|\nabla \cdot \xi\|_{H^{s_2+\frac{3}{2}}} \\ &\lesssim \|\nabla \cdot \xi^m\|_{H^{s_2+\frac{1}{2}-\varepsilon}} \|\xi\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m\|_{H^{s_2+\frac{1}{2}-\varepsilon}} \|\nabla \cdot \xi\|_{H^{s_2+\frac{3}{2}}} \\ &\lesssim \|\xi^m\|_{H^{s_2+\frac{3}{2}-\varepsilon_0}} \|\xi\|_{H^{s_2+\frac{5}{2}}}, \end{aligned}$$

where we have used $s_1 - (s_2 + \frac{3}{2}) + \frac{d}{2} \leq \frac{d}{2} \leq s_1 - 1 = s_2 + \frac{1}{2} - \varepsilon$, and $0 < \varepsilon_0 < \varepsilon$.

Inserting this inequality and (3.4) into (3.3), we get

$$(3.6) \quad \begin{aligned} \|u^{m+1}\|_{\tilde{L}_t^\infty(H^{s_1})} &\lesssim \exp(C \int_0^t \|u^m\|_{H^{s_1}} dt') \left(\|u_0\|_{H^{s_1}} + \|u^m\|_{\tilde{L}_t^\infty(H^{s_1})}^2 t \right. \\ &\quad \left. + \|\xi^m\|_{\tilde{L}^{\frac{4}{1-2\varepsilon_0}}(H^{s_2+1+\frac{2}{1-2\varepsilon_0}})} \|\xi^m\|_{\tilde{L}^{\frac{4}{3}}(H^{s_2+1+\frac{2}{3}})} t^{\frac{\varepsilon_0}{2}} \right). \end{aligned}$$

As regards z^{m+1} , it follows from Lemma 2.15 that

$$\begin{aligned} &\|z^{m+1}\|_{\tilde{L}_t^\infty(H^{s_2})} + \|z^{m+1}\|_{\tilde{L}_t^1(H^{s_2+2})} \\ &\lesssim (1+t) \left(\|z_0\|_{H^{s_2}} + \|\nabla \cdot (u^m z^m)\|_{\tilde{L}_t^1(H^{s_2})} + \|\nabla \cdot ((\nabla \cdot \xi^m)\xi^m)\|_{\tilde{L}_t^1(H^{s_2})} \right). \end{aligned}$$

According to Lemma 2.12, we get

$$(3.7) \quad \begin{aligned} \|\nabla \cdot (u^m z^m)\|_{\tilde{L}_t^1(H^{s_2})} &\lesssim \|u^m z^m\|_{\tilde{L}_t^1(H^{s_2+1})} \lesssim \|u^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} \|z^m\|_{\tilde{L}_t^2(H^{s_2+1})} t^{\frac{1}{2}} \\ &\lesssim \|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} \|z^m\|_{\tilde{L}_t^2(H^{s_2+1})} t^{\frac{1}{2}}, \end{aligned}$$

$$(3.8) \quad \begin{aligned} \|\nabla \cdot ((\nabla \cdot \xi^m)\xi^m)\|_{\tilde{L}_t^1(H^{s_2})} &\lesssim \|(\nabla \cdot \xi^m)\xi^m\|_{\tilde{L}_t^1(H^{s_2+1})} \lesssim \|(\nabla \cdot \xi^m)\|_{\tilde{L}_t^2(H^{s_2+1})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}} \\ &\lesssim \|\xi^m\|_{\tilde{L}_t^2(H^{s_2+2})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}}. \end{aligned}$$

Thus, we conclude that

$$(3.9) \quad \|z^{m+1}\|_{\tilde{L}_t^\infty(H^{s_2})} + \|z^{m+1}\|_{\tilde{L}_t^1(H^{s_2+2})}$$

$$\lesssim (1+t) \left(\|z_0\|_{H^{s_2}} + \|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} \|z^m\|_{\tilde{L}_t^2(H^{s_2+1})} t^{\frac{1}{2}} + \|\xi^m\|_{\tilde{L}_t^2(H^{s_2+2})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}} \right).$$

Similarly, combining Lemma 2.12 with Lemma 2.15 yields

$$\begin{aligned} (3.10) \quad & \|\xi^{m+1}\|_{\tilde{L}_t^\infty(H^{s_2+1})} + \|\xi^{m+1}\|_{\tilde{L}_t^1(H^{s_2+3})} \\ & \lesssim (1+t) \left(\|\xi_0\|_{H^{s_2}} + \|\mathcal{L}(u^m(\nabla \cdot \xi^m))\|_{\tilde{L}_t^1(H^{s_2+1})} + \|\mathcal{L}(z^m \xi^m)\|_{\tilde{L}_t^1(H^{s_2+1})} \right) \\ & \lesssim (1+t) \left(\|\xi_0\|_{H^{s_2}} + \|u^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} \|\nabla \cdot \xi^m\|_{\tilde{L}_t^2(H^{s_2+1})} t^{\frac{1}{2}} + \|z^m\|_{\tilde{L}_t^2(H^{s_2+1})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}} \right) \\ & \lesssim (1+t) \left(\|\xi_0\|_{H^{s_2}} + \|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} \|\xi^m\|_{\tilde{L}_t^2(H^{s_2+2})} t^{\frac{1}{2}} + \|z^m\|_{\tilde{L}_t^2(H^{s_2+1})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}} \right). \end{aligned}$$

Denote

$$E^m(t) \triangleq \|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} + \|z^m\|_{\tilde{L}_t^\infty(H^{s_2}) \cap \tilde{L}_t^1(H^{s_2+2})} + \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1}) \cap \tilde{L}_t^1(H^{s_2+3})},$$

and

$$E^0 \triangleq \|u_0\|_{H^{s_1}} + \|z_0\|_{H^{s_2}} + \|\xi_0\|_{H^{s_2+1}}.$$

By using interpolation and plugging the inequalities (3.9) and (3.10) into (3.6) yield

$$E^{m+1}(t) \leq C \left(e^{CE^m(t)t} + 1 + t \right) \left(E^0 + (E^m(t))^2 \left(t + t^{\frac{1}{2}} + t^{\frac{\varepsilon_0}{2}} \right) \right).$$

Let us choose a positive $T_0 \leq 1$ such that $\exp(8C^2 E^0 T_0) \leq 2$ and $T_0^{\frac{\varepsilon_0}{2}} \leq \frac{1}{192C^2 E^0}$. The induction hypothesis then implies that

$$E^m(T_0) \leq 8CE^0.$$

3.1.2 Second step: Convergence of the sequence

Let us fix some positive T such that $T \leq T_0$, and $(2CE^0)^4 T \leq 1$. We first consider the case $s_1 \neq 2 + \frac{d}{2}$.

By taking the difference between the equations for u^{m+1} and u^m , one finds that

$$\begin{aligned} (3.11) \quad & (u^{m+1} - u^m)_t + u^m \cdot \nabla(u^{m+1} - u^m) \\ & = (u^{m-1} - u^m) \nabla u^m - \Pi(u^m - u^{m-1}, u^m + u^{m-1}) \\ & \quad + \mathcal{P}((\nabla \cdot \xi^m)(\xi^m - \xi^{m-1})) + \mathcal{P}((\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1})\xi^{m-1}). \end{aligned}$$

Thanks to Lemma (2.17), we have

$$\begin{aligned} (3.12) \quad & \|\Pi(u^m - u^{m-1}, u^m + u^{m-1})\|_{\tilde{L}_t^1(H^{s_1-1})} \\ & \lesssim \|u^{m-1} - u^m\|_{\tilde{L}_t^\infty(H^{s_1-1})} (\|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} + \|u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_1})}) t. \end{aligned}$$

From Lemmas 2.10-2.11, we deduce that

$$(3.13) \quad \|(u^{m-1} - u^m) \nabla u^{m+1}\|_{\tilde{L}_t^1(H^{s_1-1})} \lesssim \|u^{m-1} - u^m\|_{\tilde{L}_t^\infty(H^{s_1-1})} \|u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_1})} t,$$

$$(3.14) \quad \|\mathcal{P}((\nabla \cdot \xi^m)(\xi^m - \xi^{m-1}))\|_{H^{s_1-1}}$$

$$\begin{aligned}
&\lesssim \|((\nabla \cdot \xi^m)(\xi^m - \xi^{m-1}))\|_{H^{s_1-1}} \\
&\lesssim \|\nabla \cdot \xi^m\|_{B_{\infty,\infty}^{s_1-1-(s_2+\frac{3}{2})}} \|\xi^m - \xi^{m-1}\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m - \xi^{m-1}\|_{B_{\infty,\infty}^{s_1-(s_2+\frac{3}{2})}} \|\nabla \cdot \xi^m\|_{H^{s_2+\frac{1}{2}}} \\
&\lesssim \|\nabla \cdot \xi^m\|_{H^{s_1-1-(s_2+\frac{3}{2})+\frac{2}{d}}} \|\xi^m - \xi^{m-1}\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m - \xi^{m-1}\|_{H^{s_1-(s_2+\frac{3}{2})+\frac{2}{d}}} \|\nabla \cdot \xi^m\|_{H^{s_2+\frac{1}{2}}} \\
&\lesssim \|\nabla \cdot \xi^m\|_{H^{s_2-\frac{1}{2}-\varepsilon_0}} \|\xi^m - \xi^{m-1}\|_{H^{s_2+\frac{3}{2}}} + \|\xi^m - \xi^{m-1}\|_{H^{s_2+\frac{1}{2}-\varepsilon_0}} \|\nabla \cdot \xi^m\|_{H^{s_2+\frac{1}{2}}},
\end{aligned}$$

where we have used $s_1 - 1 - (s_2 + \frac{3}{2}) + \frac{2}{d} \leq \frac{2}{d} - 1 \leq s_1 - 2 \leq s_2 + \frac{1}{2} - \varepsilon$. Similarly,

$$\begin{aligned}
(3.15) \quad &\|\mathcal{P}((\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1})\xi^{m-1})\|_{H^{s_1-1}} \\
&\lesssim \|\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1}\|_{H^{s_2-\frac{1}{2}-\varepsilon_0}} \|\xi^{m-1}\|_{H^{s_2+\frac{3}{2}}} + \|\xi^{m-1}\|_{H^{s_2+\frac{1}{2}-\varepsilon_0}} \|\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1}\|_{H^{s_2+\frac{1}{2}}}.
\end{aligned}$$

Applying Lemma 2.14 to (3.11) thus yields

$$\begin{aligned}
(3.16) \quad &\|u^{m+1} - u^m\|_{\tilde{L}_t^\infty(H^{s_1-1})} \lesssim \exp(C \int_0^t \|u^m\|_{H^{s_1}} dt') \\
&\times \left(\|u^{m-1} - u^m\|_{\tilde{L}_t^\infty(H^{s_1-1})} (\|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} + \|u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_1})}) t \right. \\
&\quad + (\|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} + \|\xi^{m-1}\|_{\tilde{L}_t^\infty(H^{s_2+1})}) \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^{\frac{4}{3}}(H^{s_2+\frac{2}{3}})} t^{\frac{1}{4}} \\
&\quad \left. + \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^{\frac{4}{1-2\varepsilon_0}}(B^{s_2+\frac{2}{1-2\varepsilon_0}})} (\|\xi^m\|_{\tilde{L}_t^4(H^{s_2+1+\frac{2}{4}})} + \|\xi^{m-1}\|_{\tilde{L}_t^4(H^{s_2+1+\frac{2}{4}})}) t^{\frac{1+\varepsilon_0}{2}} \right).
\end{aligned}$$

Note that

$$\begin{aligned}
&(z^{m+1} - z^m)_t - \Delta(z^{m+1} - z^m) = -\nabla \cdot (u^m(z^m - z^{m-1}) - (u^m - u^{m-1})z^{m-1}) \\
&\quad + \nabla((\nabla \cdot \xi^m)(\xi^m - \xi^{m-1})) - \nabla((\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1})\xi^{m-1}).
\end{aligned}$$

By virtue of Lemma 2.12, we get

$$\begin{aligned}
(3.17) \quad &\|\nabla \cdot (u^m(z^m - z^{m-1}))\|_{H^{s_2-1}} \lesssim \|u^m(z^m - z^{m-1})\|_{H^{s_2}} \\
&\lesssim \|u^m\|_{H^{s_2+\frac{1}{2}}} \|z^m - z^{m-1}\|_{H^{s_2}} \lesssim \|u^m\|_{H^{s_1}} \|z^m - z^{m-1}\|_{H^{s_2}},
\end{aligned}$$

$$\begin{aligned}
(3.18) \quad &\|\nabla \cdot (u^m - u^{m-1})z^{m-1}\|_{H^{s_2-1}} \lesssim \|(u^m - u^{m-1})z^{m-1}\|_{H^{s_2}} \\
&\lesssim \|u^m - u^{m-1}\|_{H^{s_2}} \|z^m\|_{H^{s_2+\frac{1}{2}}} \lesssim \|u^m - u^{m-1}\|_{H^{s_1-1}} \|z^m\|_{H^{s_2+\frac{1}{2}}},
\end{aligned}$$

$$\begin{aligned}
(3.19) \quad &\|\nabla((\nabla \cdot \xi^m)(\xi^m - \xi^{m-1}))\|_{H^{s_2-1}} \lesssim \|(\nabla \cdot \xi^m)(\xi^m - \xi^{m-1})\|_{H^{s_2}} \\
&\lesssim \|\nabla \cdot \xi^m\|_{H^{s_2+\frac{1}{2}}} \|\xi^m - \xi^{m-1}\|_{H^{s_2}},
\end{aligned}$$

$$\begin{aligned}
(3.20) \quad &\|\nabla((\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1})\xi^{m-1})\|_{H^{s_2-1}} \lesssim \|(\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1})\xi^{m-1}\|_{H^{s_2}} \\
&\lesssim \|\nabla \cdot \xi^m - \nabla \cdot \xi^{m-1}\|_{H^{s_2+\frac{1}{2}}} \|\xi^{m-1}\|_{H^{s_2}}.
\end{aligned}$$

Hence Lemma 2.15 implies that

$$(3.21) \quad \|z^{m+1} - z^m\|_{\tilde{L}_t^\infty(H^{s_2-1})} + \|z^{m+1} - z^m\|_{\tilde{L}_t^1(H^{s_2+1})}$$

$$\begin{aligned} &\lesssim (1+t) \left(\|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} \|z^m - z^{m-1}\|_{\tilde{L}_t^2(H^{s_2})} t^{\frac{1}{2}} + \|u^m - u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_1-1})} \|z^m\|_{\tilde{L}_t^4(H_{p_2, r_2}^{s_2+\frac{2}{4}})} t^{\frac{3}{4}} \right. \\ &\quad \left. + \|\xi^m\|_{\tilde{L}_t^4(H^{s_2+1+\frac{2}{4}})} \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^\infty(H^{s_2})} t^{\frac{3}{4}} + \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^{\frac{4}{3}}(H_{p_2, r_2}^{s_2+\frac{2}{3}})} \|\xi^{m-1}\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{4}} \right). \end{aligned}$$

Similarly, we get

$$\begin{aligned} (3.22) \quad &\|\xi^{m+1} - \xi^m\|_{\tilde{L}_t^\infty(H^{s_2})} + \|\xi^{m+1} - \xi^m\|_{\tilde{L}_t^1(H^{s_2+2})} \\ &\lesssim (1+t) \left(\|\mathcal{L}(u^m \nabla \cdot (\xi^m - \xi^{m-1}))\|_{\tilde{L}_t^1(H^{s_2})} + \|\mathcal{L}((u^m - u^{m-1}) \nabla \cdot \xi^{m-1})\|_{\tilde{L}_t^1(H^{s_2})} \right. \\ &\quad \left. + \|\mathcal{L}(z^m(\xi^m - \xi^{m-1}))\|_{\tilde{L}_t^1(H^{s_2})} + \|\mathcal{L}((z^m - z^{m-1})\xi^m)\|_{\tilde{L}_t^1(H^{s_2})} \right) \\ &\lesssim (1+t) \left(\|u^m\|_{\tilde{L}_t^\infty(H^{s_2+\frac{1}{2}})} \|\nabla \cdot (\xi^m - \xi^{m-1})\|_{\tilde{L}_t^2(H^{s_2})} t^{\frac{1}{2}} + \|u^m - u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_2})} \|\nabla \cdot \xi^{m-1}\|_{\tilde{L}_t^4(H^{s_2+\frac{2}{4}})} t^{\frac{3}{4}} \right. \\ &\quad \left. + \|z^m\|_{\tilde{L}_t^\infty(H^{s_2})} \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^4(H^{s_2+\frac{2}{4}})} t^{\frac{3}{4}} + \|z^m - z^{m-1}\|_{\tilde{L}_t^2(H^{s_2})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+\frac{1}{2}})} t^{\frac{1}{2}} \right) \\ &\lesssim (1+t) \left(\|u^m\|_{\tilde{L}_t^\infty(H^{s_1})} \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^2(H^{s_2+1})} t^{\frac{1}{2}} + \|u^m - u^{m-1}\|_{\tilde{L}_t^\infty(H^{s_1-1})} \|\xi^{m-1}\|_{\tilde{L}_t^4(H^{s_2+1+\frac{2}{4}})} t^{\frac{3}{4}} \right. \\ &\quad \left. + \|z^m\|_{\tilde{L}_t^\infty(H^{s_2})} \|\xi^m - \xi^{m-1}\|_{\tilde{L}_t^4(H^{s_2+\frac{2}{4}})} t^{\frac{3}{4}} + \|z^m - z^{m-1}\|_{\tilde{L}_t^2(H^{s_2})} \|\xi^m\|_{\tilde{L}_t^\infty(H^{s_2+1})} t^{\frac{1}{2}} \right). \end{aligned}$$

Denote

$$\begin{aligned} F^m(t) &\triangleq \|u^{m+1} - u^m\|_{\tilde{L}_t^\infty(H^{s_1-1})} + \|z^{m+1} - z^m\|_{\tilde{L}_t^\infty(H^{s_2-1}) \cap \tilde{L}_t^1(H^{s_2+1})} \\ &\quad + \|\xi^{m+1} - \xi^m\|_{\tilde{L}_t^\infty(H^{s_2}) \cap \tilde{L}_t^1(H^{s_2+2})}. \end{aligned}$$

Plugging the inequalities (3.21) and (3.22) into (3.16) yields

$$\begin{aligned} F^{m+1}(T) &\leq C(e^{CE^m(T)T} + 1 + T)(E^m(T) + E^{m-1}(T))F^m(T)(T + T^{\frac{1+\varepsilon_0}{2}} + T^{\frac{1}{2}} + T^{\frac{3}{4}} + T^{\frac{1}{4}}) \\ &\leq CE^0 T^{\frac{1}{4}} F^m(T) \leq \frac{1}{2} F^m(T). \end{aligned}$$

Hence, $(u^m, z^m, \xi^m)|_{m \in \mathbb{N}}$ is a Cauchy sequence in $\tilde{L}_T^\infty(H^{s_1-1}) \times (\tilde{L}_T^\infty(H^{s_2-1}) \cap \tilde{L}_T^1(H^{s_2+1})) \times (\tilde{L}_T^\infty(H^{s_2}) \cap \tilde{L}_T^1(H^{s_2+2}))$.

In the case $s_1 = 2 + \frac{d}{2}$, for every $\zeta \in (0, 1)$, we have

$$s_1 - \zeta > 1 + \frac{d}{2}, \text{ and } s_2 - \zeta + \frac{3}{2} > s_1 - \zeta \geq s_2 - \zeta + 1.$$

Following along the same lines as above, we have $(u^m, z^m, \xi^m)|_{m \in \mathbb{N}}$ is a Cauchy sequence in $\tilde{L}_T^\infty(H^{s_1-\zeta-1}) \times (\tilde{L}_T^\infty(H^{s_2-\zeta-1}) \cap \tilde{L}_T^1(H^{s_2-\zeta+1})) \times (\tilde{L}_T^\infty(H^{s_2-\zeta}) \cap \tilde{L}_T^1(H^{s_2-\zeta+2}))$.

3.1.3 Third step: Passing to the limit

Since the case $s_1 = 2 + \frac{d}{2}$ works the same way, we only consider the case $s_1 \neq 2 + \frac{d}{2}$. Let (u, z, ξ) be the limit of the sequence $(u^m, z^m, \xi^m)|_{m \in \mathbb{N}}$. We see that $(u, z, \xi) \in \tilde{L}_T^\infty(H^{s_1-1}) \times (\tilde{L}_T^\infty(H^{s_2-1}) \cap \tilde{L}_T^1(H^{s_2+1})) \times (\tilde{L}_T^\infty(H^{s_2}) \cap \tilde{L}_T^1(H^{s_2+2}))$. Using Lemma 2.5 with the uniform bounds given in Step 1, we see that $(u, n, p) \in \tilde{L}_T^\infty(H^{s_1}) \times (\tilde{L}_T^\infty(H^{s_2})) \times (\tilde{L}_T^\infty(H^{s_2+1}))$. Next, by interpolating we discover that (u^m, z^m, ξ^m) tends to (u, z, ξ) in every space $\tilde{L}_T^\infty(H^{s_1-\eta}) \times (\tilde{L}_T^\infty(H^{s_2-\eta}) \cap \tilde{L}_T^1(H^{s_2+1})) \times (\tilde{L}_T^\infty(H^{s_2+1-\eta}) \cap \tilde{L}_T^1(H^{s_2+2}))$, with $\eta > 0$, which suffices to pass to the limit in the system (03).

We still have to prove that $(z, \xi) \in \tilde{L}_T^1(H^{s_2+2}) \times \tilde{L}_T^1(H^{s_2+3})$. In fact, it is easy to check that $(\partial_t z - \Delta z, \partial_t \xi - \Delta \xi) \in \tilde{L}_T^1(H^{s_2}) \times \tilde{L}_T^1(H^{s_2+1})$. Hence according to Lemma 2.15, $(z, \xi) \in \tilde{L}_T^1(H^{s_2+2}) \times \tilde{L}_T^1(H^{s_2+3})$.

Finally, following along the same lines as in Theorem 3.19 of [1], we can show that

$$(3.23) \quad (u, z, \xi) \in C([0, T]; H^{s_1}) \times C([0, T]; H^{s_2}) \times C([0, T]; H^{s_2+1}).$$

3.2. Uniqueness for the system (03)

Without loss of generality, we may assume that $s_1 < 2 + \frac{d}{2}$. Assume that we are given (u_1, z_1, ξ_1) and (u_2, z_2, ξ_2) , two solutions of the system (03) (with the same initial data) satisfying the regularity assumptions of Theorem 1.1.

In order to show these two solutions coincide, we first denote

$$E^i(T) \triangleq \|u_i\|_{\tilde{L}_T^\infty(H^{s_1})} + \|z_i\|_{\tilde{L}_T^\infty(H^{s_2})} + \|\xi_i\|_{\tilde{L}_t^\infty(H^{s_2+1})}, \quad i = 1, 2,$$

$$F(t_1, t_2) \triangleq \|u_2 - u_1\|_{\tilde{L}_{[t_1, t_2]}^\infty(H^{s_1-1})} + \|z_2 - z_1\|_{\tilde{L}_{[t_1, t_2]}^\infty(H^{s_2-1})} + \|\xi_2 - \xi_1\|_{\tilde{L}_{[t_1, t_2]}^\infty(H^{s_2})},$$

and

$$T_0 = \sup\{0 \leq T' \leq T \mid (u_1, z_1, \xi_1) = (u_2, z_2, \xi_2) \text{ on } [0, T']\}.$$

We deduce from the definition of T_0 and the continuity of (u_i, z_i, ξ_i) that

$$(u_1(T_0), z_1(T_0), \xi_1(T_0)) = (u_2(T_0), z_2(T_0), \xi_2(T_0)).$$

If $T_0 < T$, repeating the same arguments as we were used for the proof of the convergence of the approximate solutions in the above subsection, we get

$$F(T_0 + \tilde{T}) \leq C(e^{CE^2(T)\tilde{T}} + 1 + \tilde{T})(E^2(T) + E^1(T))F(T_0 + \tilde{T})(\tilde{T} + \tilde{T}^{\frac{1+\varepsilon_0}{2}} + \tilde{T}^{\frac{1}{2}} + \tilde{T}^{\frac{3}{4}} + \tilde{T}^{\frac{1}{4}}).$$

We conclude that $F(T_0 + \tilde{T}) = 0$ with sufficiently small \tilde{T} . Thus, $(u_1, z_1, \xi_1) = (u_2, z_2, \xi_2)$ on $[T_0, T_0 + \tilde{T}]$, which stands in contradiction to the definition of T_0 . Hence $T_0 = T$, and the proof of uniqueness is completed.

3.3. Properties of (u, z, ξ)

3.3.1 $\nabla \cdot u = 0$

Suppose that (u, z, ξ) satisfies the system (03) in $\tilde{L}_T^\infty(H^{s_1}(\mathbb{R}^d)) \times (\tilde{L}_T^\infty(H^{s_2}(\mathbb{R}^d)) \cap \tilde{L}_T^1(H^{s_2+2}(\mathbb{R}^d))) \times (\tilde{L}_T^\infty(H^{s_2+1}(\mathbb{R}^d)) \cap \tilde{L}_T^1(H^{s_2+3}(\mathbb{R}^d)))$. We check that u is divergence free. This may be achieved by applying $\nabla \cdot$ to the first equation of the system (03). Denote $s'_1 = s_1 - 1$, if $s_1 \neq 2 + \frac{d}{2}$; and $s'_1 = s_1 - \zeta - 1$, for some $\zeta \in (0, 1)$, if $s_1 = 2 + \frac{d}{2}$. We get

$$(\partial_t + u \cdot \nabla)(\nabla \cdot u) = -\nabla \cdot \Pi(u, u) - \text{tr}(Du)^2.$$

Lemma 2.14 and Lemma 2.18 ensure that

$$\|\nabla \cdot u\|_{H^{s'_1}} \lesssim \int_0^t \exp\left(C \int_{t'}^t \|u\|_{H^{s_1}} dt''\right) \|\nabla \cdot \Pi(u, u) + \text{tr}(Du)^2\|_{H^{s'_1}} dt'$$

$$\begin{aligned}
&\lesssim \int_0^t \exp\left(C \int_{t'}^t \|u\|_{H^{s_1}} dt''\right) \|\nabla \cdot u\|_{B_{\infty,\infty}^0} \|u\|_{H^{s'_1+1}} dt', \\
&\lesssim \int_0^t \exp\left(C \int_{t'}^t \|u\|_{H^{s_1}} dt''\right) \|\nabla \cdot u\|_{H^{s'_1}} \|u\|_{H^{s_1}} dt',
\end{aligned}$$

where we have used $H^{s'_1} \hookrightarrow B_{\infty,\infty}^0$, and $H^{s'_1+1} \hookrightarrow H^{s_1}$. Using Gronwall's inequality, we conclude that $\nabla \cdot u = 0$.

3.3.2 $\mathcal{L}\xi = \xi$

Note that

$$(3.24) \quad \xi = e^{t\Delta} \xi_0 - \int_0^t e^{(t-t')\Delta} \mathcal{L}(u(\nabla \cdot \xi) + z\xi) dt'.$$

Applying \mathcal{L} to the above equation yields

$$\mathcal{L}\xi = e^{t\Delta} \mathcal{L}\xi_0 - \int_0^t e^{(t-t')\Delta} \mathcal{L}^2(u(\nabla \cdot \xi) + z\xi) dt'.$$

It is easy to check that

$$\begin{aligned}
\mathcal{L}\xi_0 &= -\nabla(-\Delta)^{-1} \nabla \cdot (-\nabla(-\Delta)^{-1} a_0) = \nabla(-\Delta)^{-1} (\nabla \cdot \nabla) (-\Delta)^{-1} a_0 = -\nabla(-\Delta)^{-1} a_0 = \xi_0, \\
\mathcal{L}^2 &= -\nabla(-\Delta)^{-1} \nabla \cdot (-\nabla(-\Delta)^{-1} \nabla \cdot) = \nabla(-\Delta)^{-1} (\nabla \cdot \nabla) (-\Delta)^{-1} \nabla \cdot = -\nabla(-\Delta)^{-1} \nabla \cdot = \mathcal{L}.
\end{aligned}$$

Hence, $\mathcal{L}\xi = \xi$.

3.3.3 Nonnegative of $z \pm \nabla \cdot \xi$

Let $a = \frac{z+\nabla \cdot \xi}{2}$, and $b = \frac{z-\nabla \cdot \xi}{2}$. As $\nabla \cdot \mathcal{L} = \nabla \cdot$ and $\nabla \cdot u = 0$, one finds that (a, b) solves the following system:

$$(ab) \quad \begin{cases} a_t + u \cdot \nabla a - \Delta a = -\nabla \cdot (a\xi), \\ b_t + u \cdot \nabla b - \Delta b = \nabla \cdot (b\xi), \\ \nabla \cdot \xi = a - b, \\ (a, b)|_{t=0} = \left(\frac{z_0+\nabla \cdot \xi_0}{2}, \frac{z_0-\nabla \cdot \xi_0}{2}\right), \end{cases}$$

We test the first equation of the system (ab) with $(a^-) \triangleq \sup\{-a, 0\}$. After integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|a^-\|_{L^2}^2 + \|\nabla a^-\|_{L^2}^2 = - \int_{\mathbb{R}^d} \frac{1}{2} (\nabla \cdot \xi) (a^-)^2 dx \leq \frac{1}{2} \|\nabla \cdot \xi\|_{L^\infty} \|a^-\|_{L^2}^2.$$

Gronwall's Lemma implies that

$$\|a^-\|_{L^2}^2 \leq \|a_0^-\|_{L^2}^2 \exp\{\|\nabla \cdot \xi\|_{L_t^1(L^\infty)}\}.$$

Since $a_0 \geq 0$, and

$$\|\nabla \cdot \xi\|_{L_t^1(L^\infty)} \lesssim \|\nabla \cdot \xi\|_{L_t^1(H^{s_2+\frac{1}{2}})} \lesssim \|\nabla \cdot \xi\|_{L_t^2(H^{s_2+1})} t^{\frac{1}{2}},$$

we have $\|a^-\|_{L^2}^2 = 0$. Hence $a \geq 0$, a.e. on $[0, T] \times \mathbb{R}^d$. Repeating the same steps for b implies $b \geq 0$, a.e. on $[0, T] \times \mathbb{R}^d$, and thus $z = a + b \geq 0$, a.e. on $[0, T] \times \mathbb{R}^d$.

3.4. A global existence result in dimension $d = 2$

According to the above subsections, local existence in $\widetilde{L}_T^\infty(H^{s_1}(\mathbb{R}^d)) \times (\widetilde{L}_T^\infty(H^{s_2}(\mathbb{R}^d)) \cap \widetilde{L}_T^1(H^{s_2+2}(\mathbb{R}^d))) \times (\widetilde{L}_T^\infty(H^{s_2+1}(\mathbb{R}^d)) \cap \widetilde{L}_T^1(H^{s_2+3}(\mathbb{R}^d)))$ has already been proven. So we denote by T^* the maximal time of existence of (u, z, ξ) . Suppose that T^* is finite, under the assumption of 1.1, and assume further that $d = 2$ and $s_2 > 1$, we have the following lemmas.

3.4.1 Some useful lemmas

Lemma 3.1. $\forall t \in [0, T^*)$, we have

$$(3.25) \quad \|u(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2 + \int_0^t \|\nabla \xi\|_{L^2}^2 dt' \lesssim \|u_0\|_{L^2}^2 + \|\xi_0\|_{L^2}^2.$$

Proof. Multiplying the first and the third equations of the system (03) by u and ξ respectively, and integrating over \mathbb{R}^d :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 &= \int_{\mathbb{R}^2} (\nabla \cdot \xi) \xi u dx, \\ \frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2}^2 + \int_{\mathbb{R}^2} (\nabla \cdot \xi) \xi u dx + \|\nabla \xi\|_{L^2}^2 &= \int_{\mathbb{R}^d} -z |\xi|^2 dx, \end{aligned}$$

where we have used

$$\begin{aligned} \int_{\mathbb{R}^2} (u \cdot \nabla u) u dx &= \int_{\mathbb{R}^2} -\frac{1}{2} (\nabla \cdot u) |u|^2 dx = 0, \\ \int_{\mathbb{R}^2} \Pi(u, u) u dx &= \int_{\mathbb{R}^2} \nabla P_\Pi(u, u) u dx = \int_{\mathbb{R}^2} -P_\Pi(u, u) (\nabla \cdot u) dx = 0, \\ \int_{\mathbb{R}^2} \mathcal{P}((\nabla \cdot \xi) \xi) u dx &= \int_{\mathbb{R}^2} (\nabla \cdot \xi) \xi (\mathcal{P}u) dx = \int_{\mathbb{R}^2} (\nabla \cdot \xi) \xi u dx, \\ \int_{\mathbb{R}^2} \mathcal{L}(u(\nabla \cdot \xi)) \xi dx &= \int_{\mathbb{R}^2} u (\nabla \cdot \xi) (\mathcal{L}\xi) dx = \int_{\mathbb{R}^2} u (\nabla \cdot \xi) \xi dx, \\ \int_{\mathbb{R}^2} \mathcal{L}(-z\xi) \xi &= \int_{\mathbb{R}^2} -z \xi (\mathcal{L}\xi) = \int_{\mathbb{R}^2} -z |\xi|^2, \end{aligned}$$

with $P_\pi(u, u)$ defined as in Lemma 2.16. Summing the above equations and using the fact $z \geq 0$, we find

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|\xi\|_{L^2}^2) + \|\nabla \xi\|_{L^2}^2 \leq 0,$$

from which it follows that (3.25) holds. \square

Lemma 3.2. $\forall t \in [0, T^*)$, $2 \leq q \leq \infty$, we have

$$\begin{aligned} \|z \pm \nabla \cdot \xi(t)\|_{L^q} &\lesssim \|z_0 + \nabla \cdot \xi_0\|_{L^q} + \|z_0 - \nabla \cdot \xi_0\|_{L^q}, \\ \|z \pm \nabla \cdot \xi(t)\|_{L^2}^2 + \int_0^t \|\nabla(z \pm \nabla \cdot \xi)\|_{L^2}^2 dt' &\lesssim \|z_0 + \nabla \cdot \xi_0\|_{L^2}^2 + \|z_0 - \nabla \cdot \xi_0\|_{L^2}^2. \end{aligned}$$

Proof. Since $s_2 > 1$, (3.23) implies that $(a, b) \in (C([0, T]; H^{s_2}))^2 \hookrightarrow (C([0, T]; L^q))^2$, with $2 \leq q \leq \infty$. By multiplying both sides of the first equation of the (ab) system by $|a|^{p-2}a$ with $2 \leq p < \infty$, and integrating over $[0, t] \times \mathbb{R}^d$, we get

$$(3.26) \quad \frac{1}{p} \|a(t)\|_{L^p}^p + (p-1) \int_{\mathbb{R}^2} |a|^{p-2} |\nabla a|^2 dx \leq \frac{1}{p} \|a_0\|_{L^p}^p - \frac{p-1}{p} \int_0^t \int_{\mathbb{R}^2} \nabla \cdot \xi |a|^p dx dt',$$

where we have used the estimates

$$\begin{aligned} \int_{\mathbb{R}^2} |a|^{p-2} a (u \cdot \nabla a) dx &= -\frac{1}{p} \int_{\mathbb{R}^2} (\nabla \cdot u) |a|^p dx = 0, \\ - \int_{\mathbb{R}^2} |a|^{p-2} a \nabla \cdot (a \xi) dx &= - \int_{\mathbb{R}^2} \xi \frac{1}{p} \nabla |a|^p dx - \int_{\mathbb{R}^2} (\nabla \cdot \xi) |a|^p dx \\ &= - \int_{\mathbb{R}^2} \frac{p-1}{p} (\nabla \cdot \xi) |a|^p dx. \end{aligned}$$

Repeating the same steps for b yields

$$(3.27) \quad \frac{1}{p} \|b(t)\|_{L^p}^p + (p-1) \int_{\mathbb{R}^2} |b|^{p-2} |\nabla b|^2 dx \leq \frac{1}{p} \|b_0\|_{L^p}^p + \frac{p-1}{p} \int_0^t \int_{\mathbb{R}^2} (\nabla \cdot \xi) |b|^p dx dt'.$$

Adding up (3.26) and (3.27), we get

$$\begin{aligned} &\frac{1}{p} (\|a(t)\|_{L^p}^p + \|b(t)\|_{L^p}^p) + (p-1) \int_{\mathbb{R}^2} |a|^{p-2} |\nabla a|^2 dx + (p-1) \int_{\mathbb{R}^2} |b|^{p-2} |\nabla b|^2 dx \\ &\leq \frac{1}{p} (\|a_0\|_{L^p}^p + \|b_0\|_{L^p}^p) + \frac{p-1}{p} \int_0^t \int_{\mathbb{R}^2} \nabla \cdot \xi (|b|^p - |a|^p) dx dt' \\ &\leq \frac{1}{p} (\|a_0\|_{L^p}^p + \|b_0\|_{L^p}^p) + \frac{p-1}{p} \int_0^t \int_{\mathbb{R}^2} (a-b)(b^p - a^p) dx dt' \\ &\leq \frac{1}{p} (\|a_0\|_{L^p}^p + \|b_0\|_{L^p}^p), \end{aligned}$$

where we have used the non-negativity of a, b . This thus leads to

$$\begin{aligned} \|a(t)\|_{L^p} + \|b(t)\|_{L^p} &\leq 2(\|a_0\|_{L^p} + \|b_0\|_{L^p}), \\ \|a(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t (\|\nabla a\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) dt' &\leq \|a_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned}$$

Passing to the limit as p tends to infinite gives

$$\|a(t)\|_{L^\infty} + \|b(t)\|_{L^\infty} \leq 2(\|a_0\|_{L^\infty} + \|b_0\|_{L^\infty}).$$

This completes the proof of the lemma. □

Lemma 3.3. $\forall t \in [0, T^*)$, we have

$$(3.28) \quad \|\xi(t)\|_{L^\infty} \leq C(T^*) < \infty.$$

Proof. It is easy to obtain from 3.24 that

$$\begin{aligned} \|\xi(t)\|_{L^\infty} &\lesssim \|\mathcal{F}^{-1}(e^{-t|x|^2})\|_{L^1} \|\xi_0\|_{L^\infty} + \int_0^t \|\mathcal{F}^{-1}(e^{(t'-t)|x|^2})\|_{L^2} \left(\|u(t')\|_{L^2} \|\nabla \cdot \xi(t')\|_{L^\infty} \right. \\ &\quad \left. + \|z(t')\|_{L^\infty} \|\xi(t')\|_{L^2} \right) dt' \\ &\lesssim \|\xi_0\|_{L^\infty} + \int_0^t (t'-t)^{-\frac{1}{2}} \left(\|u(t')\|_{L^2} \|\nabla \cdot \xi(t')\|_{L^\infty} dt' + \|z(t')\|_{L^\infty} \|\xi(t')\|_{L^2} \right) \\ &\lesssim \|\xi_0\|_{L^\infty} + t^{\frac{1}{2}} \left(\|u\|_{L_t^\infty(L^2)} \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} + \|z\|_{L_t^\infty(L^\infty)} \|\xi\|_{L_t^\infty(L^2)} \right). \end{aligned}$$

Applying Lemma 3.1 and Lemma 3.2 completes the proof. □

Denote $w = \partial_2 u_1 - \partial_1 u_2$. Note that $\Pi(u, u) = \nabla P_\Pi(u, u)$, where $P_\Pi(u, u)$ defined as in Lemma 2.16, $(\mathcal{P} - Id)((\nabla \cdot \xi)\xi) = \nabla(-\Delta)^{-1} \nabla \cdot ((\nabla \cdot \xi)\xi)$, and $\xi = \mathcal{L}\xi = \nabla(-\Delta)^{-1} \nabla \cdot \xi$. Then w satisfies

$$(3.29) \quad w_t + u \cdot \nabla w = \partial_2(\nabla \cdot \xi)\xi_1 - \partial_1(\nabla \cdot \xi)\xi_2.$$

Lemma 3.4. *[1] For all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, there exists a constant C such that*

$$(3.30) \quad \|(Id - \Delta_{-1})u\|_{B_{p,r}^s} \leq C\|w\|_{B_{p,r}^s}.$$

Lemma 3.5. $\forall t \in [0, T^*)$, we have

$$(3.31) \quad \|\nabla u(t)\|_{L^2} \leq C(T^*) < \infty.$$

Proof. Multiplying (3.4) by w and integrating over \mathbb{R}^2 :

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \lesssim \|\nabla(\nabla \cdot \xi)\|_{L^2} \|\xi\|_{L^\infty} \|w\|_{L^2}.$$

The Gronwall lemma implies that

$$\|w(t)\|_{L^2} \lesssim \|w_0\|_{L^2} + \int_0^t \|\nabla(\nabla \cdot \xi)\|_{L^2} \|\xi\|_{L^\infty} dt' \lesssim \|w_0\|_{L^2} + \|\nabla(\nabla \cdot \xi)\|_{L_t^2(L^2)} \|\xi\|_{L_t^\infty(L^\infty)} t^{\frac{1}{2}}.$$

Applying Lemma 3.2 and Lemma 3.3, we have

$$(3.32) \quad \|w(t)\|_{L^2} \leq C(T^*) < \infty.$$

Next by splitting u into low and high frequencies and using Lemma 3.4, we see that

$$\|\nabla u\|_{L^2} \lesssim \|\Delta_{-1} \nabla u\|_{L^2} + \|(Id - \Delta_{-1}) \nabla u\|_{L^2} \lesssim \|u\|_{L^2} + \|w\|_{L^2}.$$

Applying Lemma 3.2 and the inequality 3.32 then completes the proof of the lemma.

Lemma 3.6.

$$(3.33) \quad \int_0^{T^*} \|\nabla(z \pm \nabla \cdot \xi)\|_{L^\infty} dt' < \infty.$$

Proof. First combining Lemma 3.1 and Lemma 3.5 with the Sobolev imbedding theorem, we see that

$$(3.34) \quad u \in L_{T^*}^\infty(H^1) \hookrightarrow L_{T^*}^\infty(L^p),$$

with $2 \leq p < \infty$. Then we denote from the system (ab) that

$$\nabla a = \nabla e^{t\Delta} a_0 - \int_0^t e^{(t-t')\Delta} \nabla(u \cdot \nabla a + \xi \cdot \nabla a + a(\nabla \cdot \xi)) dt'.$$

We have

$$\begin{aligned} & \|(\nabla a)(\tau)\|_{L^\infty} \\ & \lesssim \|\mathcal{F}^{-1}(e^{-\tau|x|^2} x)\|_{L^1} \|a_0\|_{L^\infty} + \int_0^\tau \|\mathcal{F}^{-1}(e^{(t'-\tau)|x|^2} x)\|_{L^{q'}} \|u(t')\|_{L^q} \|\nabla a(t')\|_{L^\infty} dt' \end{aligned}$$

$$\begin{aligned}
& + \int_0^\tau \left\| \mathcal{F}^{-1}(e^{(t'-\tau)|x|^2} x) \right\|_{L^1} (\|\xi\|_{L^\infty} \|\nabla a\|_{L^\infty} + \|\nabla \cdot \xi\|_{L^\infty} \|a\|_{L^\infty})(t') dt' \\
& \lesssim \tau^{-\frac{1}{2}} \|a_0\|_{L^\infty} + \int_0^\tau \frac{1}{(\tau-t')^{\frac{1}{2}+\frac{1}{q}}} \|u(t')\|_{L^q} \|\nabla a(t')\|_{L^\infty} dt' \\
& + \int_0^\tau \frac{1}{(\tau-t')^{\frac{1}{2}}} (\|\xi\|_{L^\infty} \|\nabla a\|_{L^\infty} + \|\nabla \cdot \xi\|_{L^\infty} \|a\|_{L^\infty})(t') dt' \\
& \lesssim \tau^{-\frac{1}{2}} \|a_0\|_{L^\infty} + \tau^{\frac{1}{2}} \|\nabla \cdot \xi\|_{L_T^\infty(L^\infty)} \|a\|_{L_T^\infty(L^\infty)} \\
& + \int_0^\tau \left(\frac{1}{(\tau-t')^{\frac{1}{2}+\frac{1}{q}}} \|u(t')\|_{L^q} + \frac{1}{(\tau-t')^{\frac{1}{2}}} \|\xi(t')\|_{L^\infty} \right) \|\nabla a(t')\|_{L^\infty} dt' \\
& \lesssim \tau^{-\frac{1}{2}} \|a_0\|_{L^\infty} + \tau^{\frac{1}{2}} \|\nabla \cdot \xi\|_{L_T^\infty(L^\infty)} \|a\|_{L_T^\infty(L^\infty)} \\
& + \int_0^\tau \left(\delta_1 \frac{1}{(\tau-t')^{(\frac{1}{2}+\frac{1}{q})\gamma_1}} + \delta_2 \frac{1}{(\tau-t')^{\frac{1}{2}\gamma_2}} \right) \|\nabla a(t')\|_{L^\infty} dt' \\
& + \int_0^\tau (C_{\delta_1} \|u(t')\|_{L^q}^{\gamma'_1} + C_{\delta_2} \|\xi\|_{L^\infty}^{\gamma'_2}) \|\nabla a(t')\|_{L^\infty} dt',
\end{aligned}$$

with $2 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, $(\frac{1}{2} + \frac{1}{q})\gamma_1 < 1$, $\frac{1}{2}\gamma_2 < 1$. By means of the Young inequality for the time integral, we obtain,

$$\begin{aligned}
\|\nabla a\|_{L_t^1(L^\infty)} & \lesssim t^{\frac{1}{2}} \|a_0\|_{L^\infty} + t^{\frac{3}{2}} \|\nabla \cdot \xi\|_{L_T^\infty(L^\infty)} \|a\|_{L_T^\infty(L^\infty)} \\
& + \left(\delta_1 \frac{1}{1 - (\frac{1}{2} + \frac{1}{q})\gamma_1} t^{1 - (\frac{1}{2} + \frac{1}{q})\gamma_1} + \delta_2 \frac{1}{1 - \frac{1}{2}\gamma_2} t^{1 - \frac{1}{2}\gamma_2} \right) \|\nabla a\|_{L_t^1(L^{a_1})} \\
& + \int_0^t \int_0^\tau (C_{\delta_1} \|u(t')\|_{L^q}^{\gamma'_1} + C_{\delta_2} \|\xi(t')\|_{L^\infty}^{\gamma'_2}) \|\nabla a(t')\|_{L^\infty} dt' d\tau.
\end{aligned}$$

Choosing $\delta_1 \frac{1}{1 - (\frac{1}{2} + \frac{1}{q})\gamma_1} (T^*)^{1 - (\frac{1}{2} + \frac{1}{q})\gamma_1} + \delta_2 \frac{1}{1 - \frac{1}{2}\gamma_2} (T^*)^{1 - \frac{1}{2}\gamma_2} = c \frac{1}{2}$ yields

$$\begin{aligned}
\|\nabla a\|_{L_t^1(L^\infty)} & \lesssim (T^*)^{\frac{1}{2}} \|a_0\|_{L^\infty} + (T^*)^{\frac{3}{2}} \|\nabla \cdot \xi\|_{L_{T^*}^\infty(L^\infty)} \|a\|_{L_{T^*}^\infty(L^\infty)} \\
& + (C_{\delta_1} \|u(t')\|_{L_{T^*}^\infty(L^q)}^{\gamma'_1} + C_{\delta_2} \|\xi\|_{L_{T^*}^\infty(L^\infty)}^{\gamma'_2}) \int_0^t \int_0^\tau \|\nabla a(t')\|_{L^\infty} dt' d\tau.
\end{aligned}$$

Gronwall's lemma thus implies that

$$\begin{aligned}
\|\nabla a\|_{L_t^1(L^\infty)} & \leq C \left((T^*)^{\frac{1}{2}} \|a_0\|_{L^\infty} + (T^*)^{\frac{3}{2}} \|\nabla \cdot \xi\|_{L_{T^*}^\infty(L^\infty)} \|a\|_{L_{T^*}^\infty(L^\infty)} \right) \\
& \times \exp \left((C_{\delta_1} \|u(t')\|_{L_{T^*}^\infty(L^q)}^{\gamma'_1} + C_{\delta_2} \|\xi\|_{L_{T^*}^\infty(L^\infty)}^{\gamma'_2}) t \right).
\end{aligned}$$

Hence, Lemma 3.2, Lemma 3.3 and the inequality (3.34) imply that

$$\int_0^{T^*} \|\nabla a\|_{L^\infty} dt' < \infty.$$

Similar arguments for b yield

$$\int_0^{T^*} \|\nabla b\|_{L^\infty} dt' < \infty.$$

Therefore, the inequality (3.33) holds true. \square

Lemma 3.7. $\forall t \in [0, T^*)$, we have

$$(3.35) \quad \|u(t)\|_{B_{\infty, \infty}^1} \leq C(T^*) < \infty.$$

Proof. we deduce from the inequality (3.29) that

$$\|w(t)\|_{L^\infty} \lesssim \|w_0\|_{L^\infty} + \int_0^t \|\nabla(\nabla \cdot \xi)\|_{L^\infty} \|\xi\|_{L^\infty} dt' \lesssim \|w_0\|_{L^\infty} + \|\nabla(\nabla \cdot \xi)\|_{L_t^1(L^\infty)} \|\xi\|_{L_t^\infty(L^\infty)}.$$

By splitting u into low and high frequencies and using Lemma 3.4, we see that

$$\begin{aligned} \|u\|_{B_{\infty,\infty}^1} &\lesssim \|\Delta_{-1}u\|_{L^\infty} + \|(Id - \Delta_{-1})\nabla u\|_{B_{\infty,\infty}^0} \\ &\lesssim \|u\|_{L^2} + \|w\|_{B_{\infty,\infty}^0} \lesssim \|u\|_{L^2} + \|w\|_{L^\infty}. \end{aligned}$$

Applying Lemma 3.1, Lemma 3.3 and Lemma 3.6 completes the proof of the lemma. \square

3.4.2 Proof of the global existence

We now turn to the proof of the global existence. Applying Δ_j to the first equation of the system (03) yields that

$$(\partial_t + u \cdot \nabla) \Delta_j u + \Delta_j \Pi(u, u) = \Delta_j \mathcal{P}((\nabla \cdot \xi) \xi) + R_{j1},$$

with $R_{j1} = u \cdot \nabla \Delta_j u - \Delta_j(u \cdot \nabla)u \triangleq [u \cdot \nabla, \Delta_j]u$.

Taking the L^2 inner product of the above equation with $\Delta_j u$, we easily get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Delta_j u(t)\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^2} (\nabla \cdot u) |\Delta_j u|^2 dx \\ &\leq \|\Delta_j u\|_{L^2} \left(\|\Delta_j \Pi(u, u)\|_{L^2} + \|\Delta_j \mathcal{P}((\nabla \cdot \xi) \xi)\|_{L^2} + \|R_{j1}\|_{L^2} \right), \quad j \geq -1. \end{aligned}$$

Note that $\nabla \cdot u = 0$, we get

$$\|\Delta_j u(t)\|_{L^2} \leq \|\Delta_j u_0\|_{L^2} + \int_0^t \|\Delta_j \Pi(u, u)\|_{L^2} + \|\Delta_j \mathcal{P}((\nabla \cdot \xi) \xi)\|_{L^2} + \|R_{j1}\|_{L^2} dt', \quad j \geq -1.$$

Multiplying both sides of the above inequality by 2^{js_1} , taking the l^2 norm, we obtain

$$(3.36) \quad \|u\|_{\tilde{L}_t^\infty(H^{s_1})} \lesssim \|u_0\|_{H^{s_1}} + \|\Pi(u, u)\|_{\tilde{L}_t^1(H^{s_1})} + \|\mathcal{P}((\nabla \cdot \xi) \xi)\|_{\tilde{L}_t^1(H^{s_1})} + \left\| 2^{js_1} \|R_{j1}\|_{L_t^1(L^2)} \right\|_{l^2}.$$

Due to Lemma 2.13, we get

$$(3.37) \quad \left\| 2^{js_1} \|R_{j1}\|_{L_t^1(L^2)} \right\|_{l^2} \lesssim \int_0^t \left\| 2^{js_1} \|R_{j1}\|_{L^2} \right\|_{l^2} dt' \lesssim \int_0^t \|\nabla u\|_{L^\infty} \|u\|_{H^{s_1}} dt'.$$

By virtue of Lemma 2.16, we have

$$(3.38) \quad \|\Pi(u, u)\|_{\tilde{L}_t^1(H^{s_1})} \lesssim \|\Pi(u, u)\|_{L_t^1(H^{s_1})} \lesssim \int_0^t \|u\|_{C^{0,1}} \|u\|_{H^{s_1}} dt'.$$

We now focus on the term $\mathcal{P}((\nabla \cdot \xi) \xi)$. By taking advantage of Bony's decomposition and of Lemmas 2.10-2.11, we have

$$\begin{aligned} (3.39) \quad \|\mathcal{P}((\nabla \cdot \xi) \xi)\|_{\tilde{L}_t^1(H^{s_1})} &\lesssim \|(\nabla \cdot \xi) \xi\|_{\tilde{L}_t^1(H^{s_1})} \\ &\lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_1})} + \|\xi\|_{L_t^\infty(L^\infty)} \|\nabla \cdot \xi\|_{\tilde{L}_t^1(H^{s_1})} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_2+\frac{3}{2}})} + \|\xi\|_{L_t^\infty(L^\infty)} \|\nabla \cdot \xi\|_{\tilde{L}_t^1(H^{s_2+\frac{3}{2}})} \\
&\lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_2+1})}^{\frac{3}{4}} \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}^{\frac{1}{4}} + \|\xi\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_2+1})}^{\frac{1}{4}} \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}^{\frac{3}{4}} \\
&\lesssim C_\sigma (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)}^{\frac{4}{3}} + \|\xi\|_{L_t^\infty(L_t^\infty)}^4) \int_0^t \|\xi\|_{H^{s_2+1}} dt' + \sigma \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}.
\end{aligned}$$

Plugging the inequalities (3.37)-(3.39) into (3.36), we eventually get

$$\begin{aligned}
(3.40) \quad \|u\|_{\tilde{L}_t^\infty(H^{s_1})} &\lesssim \|u_0\|_{H^{s_1}} + \int_0^t (\|u\|_{C^{0,1}} \|u\|_{H^{s_1}} \\
&\quad + C_\sigma (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)}^{\frac{4}{3}} + \|\xi\|_{L_t^\infty(L_t^\infty)}^4) \|\xi\|_{H^{s_2+1}}) dt' + \sigma \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}.
\end{aligned}$$

Similarly, applying Δ_j to the second equation of the system (03) yields that

$$(\partial_t + u \cdot \nabla - \Delta) \Delta_j z = -\Delta_j \nabla \cdot ((\nabla \cdot \xi) \xi) + R_{j2},$$

with $R_{j2} = [u \cdot \nabla, \Delta_j] z$, where we have used $\nabla \cdot (uz) = u \cdot \nabla z + (\nabla \cdot u)z = u \cdot \nabla z$.

Taking the L^2 inner product of the above equation with $\Delta_j z$, we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\Delta_j z(t)\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^2} (\nabla \cdot u) |\Delta_j z|^2 dx + \|\nabla \Delta_j z\|_{L^2}^2 \\
&\leq \|\Delta_j z\|_{L^2} \left(\|\Delta_j \nabla \cdot ((\nabla \cdot \xi) \xi)\|_{L^2} + \|R_{j2}\|_{L^2} \right), \quad j \geq -1.
\end{aligned}$$

Note that $\nabla \cdot u = 0$, $\|\nabla \Delta_{-1} z\|_{L^2} \geq 0$, and by virtue of Lemma 2.7, $\|\nabla \Delta_j z\|_{L^2} \gtrsim 2^j \|\Delta_j z\|_{L^2}$, for $j \geq 0$. Therefore, we have

$$\begin{aligned}
(3.41) \quad \|\Delta_j z(t)\|_{L^2} &+ \int_0^t 2^{2j} \|\Delta_j z\|_{L^2} dt' \lesssim (1+t) (\|\Delta_j z_0\|_{L^2} + \int_0^t \|\Delta_j \nabla \cdot ((\nabla \cdot \xi) \xi)\|_{L^2} \\
&\quad + \|R_{j2}\|_{L^2} dt'), \quad j \geq -1.
\end{aligned}$$

Hence multiplying both sides of the above inequality by 2^{js_2} and taking the l^2 norm, we obtain

$$\begin{aligned}
&\|z\|_{\tilde{L}_t^\infty(H^{s_2})} + \|z\|_{\tilde{L}_t^1(H^{s_2+2})} \\
&\lesssim (1+t) (\|z_0\|_{H^{s_2}} + \|\nabla \cdot ((\nabla \cdot \xi) \xi)\|_{\tilde{L}_t^1(H^{s_2})} + \|2^{js_2} \|R_{j2}\|_{L_t^1(L^2)}\|_{l^2}).
\end{aligned}$$

In view of Lemma 2.13, we get

$$\begin{aligned}
(3.42) \quad \left\| 2^{js_2} \|R_{j2}\|_{L_t^1(L^2)} \right\|_{l^2} &\lesssim \int_0^t \left\| 2^{js_2} \|R_{j2}\|_{L^2} \right\|_{l^2} dt' \\
&\lesssim \int_0^t (\|\nabla u\|_{L^\infty} \|z\|_{H^{s_2}} + \|\nabla z\|_{L^\infty} \|\nabla u\|_{H^{s_2-1}}) dt' \\
&\lesssim \int_0^t (\|\nabla u\|_{L^\infty} \|z\|_{H^{s_2}} + \|\nabla z\|_{L^\infty} \|u\|_{H^{s_1}}) dt'.
\end{aligned}$$

According to Lemmas 2.12, we have

$$\begin{aligned}
(3.43) \quad \|\nabla \cdot ((\nabla \cdot \xi) \xi)\|_{\tilde{L}_t^1(H^{s_2})} &\lesssim \|(\nabla \cdot \xi) \xi\|_{\tilde{L}_t^1(H^{s_2+1})} \\
&\lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_2+1})} + \|\xi\|_{L_t^\infty(L^\infty)} \|\nabla \cdot \xi\|_{\tilde{L}_t^1(H^{s_2+1})},
\end{aligned}$$

$$\lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \int_0^t \|\xi\|_{H^{s_2+1}} dt' + C_\sigma \|\xi\|_{L_t^\infty(L^\infty)}^2 \int_0^t \|\xi\|_{H^{s_2+1}} dt' + \sigma \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}.$$

Inserting the inequalities (3.42)-(3.43) into (3.41), we finally get

$$(3.44) \quad \begin{aligned} & \|z\|_{\tilde{L}_t^\infty(H^{s_2})} + \|z\|_{\tilde{L}_t^1(H^{s_2+2})} \\ & \lesssim (1+t) \left(\|z_0\|_{H^{s_2}} + \int_0^t \left(\|\nabla u\|_{L^\infty} \|z\|_{H^{s_2}} + \|\nabla z\|_{L^\infty} \|u\|_{H^{s_1}} \right. \right. \\ & \quad \left. \left. + (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} + C_\sigma \|\xi\|_{L_t^\infty(L^\infty)}^2) \|\xi\|_{H^{s_2+1}} \right) dt' + \sigma \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})} \right) \end{aligned}$$

To deal with the third equation of the system (03), we have

$$(3.45) \quad \begin{aligned} & \|\mathcal{L}(u(\nabla \cdot \xi))\|_{\tilde{L}_t^1(H^{s_2+1})} \lesssim \|u(\nabla \cdot \xi)\|_{\tilde{L}_t^1(H^{s_2+1})} \\ & \lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|u\|_{\tilde{L}_t^1(H^{s_2+1})} + \|u\|_{\tilde{L}_t^\infty(L^q)} \|\nabla \cdot \xi\|_{\tilde{L}_t^1(B^{\frac{s_2+1}{q-2}, 2})} \\ & \lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \|u\|_{\tilde{L}_t^1(H^{s_1})} + \|u\|_{L_t^\infty(L^q)} \|\nabla \cdot \xi\|_{\tilde{L}_t^1(H^{s_2+1+\frac{2}{q}})} \\ & \lesssim \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \int_0^t \|u\|_{H^{s_1}} dt' + C_\sigma \|u\|_{L_t^\infty(L^q)}^{\frac{2q}{q-2}} \int_0^t \|\xi\|_{H^{s_2+1}} dt' + \sigma \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}, \end{aligned}$$

with $2 \leq q < \infty$.

$$(3.46) \quad \begin{aligned} & \|\mathcal{L}(z\xi)\|_{\tilde{L}_t^1(H^{s_2+1})} \lesssim \|z\xi\|_{\tilde{L}_t^1(H^{s_2+1})} \\ & \lesssim \|\xi\|_{L_t^\infty(L^\infty)} \|z\|_{\tilde{L}_t^1(H^{s_2+1})} + \|z\|_{L_t^\infty(L^\infty)} \|\xi\|_{\tilde{L}_t^1(H^{s_2+1})} \\ & \lesssim C_\sigma \|\xi\|_{L_t^\infty(L^\infty)}^2 \int_0^t \|z\|_{H^{s_2}} dt' + \sigma \|z\|_{\tilde{L}_t^1(H^{s_2+2})} + \|z\|_{L_t^\infty(L^\infty)} \int_0^t \|\xi\|_{H^{s_2+1}} dt'. \end{aligned}$$

Hence,

$$(3.47) \quad \begin{aligned} & \|\xi\|_{\tilde{L}_t^\infty(H^{s_2+1})} + \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})} \\ & \lesssim (1+t) \left(\|\xi_0\|_{H^{s_2+1}} + \|\mathcal{L}(u(\nabla \cdot \xi))\|_{\tilde{L}_t^1(H^{s_2+1})} + \|\mathcal{L}(z\xi)\|_{\tilde{L}_t^1(H^{s_2+1})} \right) \\ & \lesssim (1+t) \left(\|\xi_0\|_{H^{s_2+1}} + (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} + C_\sigma \|u\|_{L_t^\infty(L^q)}^{\frac{2q}{q-2}} + C_\sigma \|\xi\|_{L_t^\infty(L^\infty)}^2 + \|z\|_{L_t^\infty(L^\infty)}) \right. \\ & \quad \left. \times \int_0^t (\|u\|_{H^{s_1}} + \|\xi\|_{H^{s_2+1}} + \|z\|_{H^{s_2}}) dt' + \sigma (\|\xi\|_{\tilde{L}_t^1(H^{s_2+3})} + \|z\|_{\tilde{L}_t^1(H^{s_2+2})}) \right). \end{aligned}$$

Combining (3.44), (3.47) and (3.40), we get $\forall t \in [0, T^*)$,

$$(3.48) \quad \begin{aligned} & \|u\|_{\tilde{L}_t^\infty(H^{s_1})} + \|z\|_{\tilde{L}_t^\infty(H^{s_2}) \cap \tilde{L}_t^1(H^{s_2+2})} + \|\xi\|_{\tilde{L}_t^\infty(H^{s_2+1}) \cap \tilde{L}_t^1(H^{s_2+3})} \\ & \lesssim (1+t) (\|u_0\|_{H^{s_1}} + \|z_0\|_{H^{s_2}} + \|\xi_0\|_{H^{s_2+1}}) + (1+t) \int_0^t \left(\|u\|_{C^{0,1}} + \|\nabla z\|_{L^\infty} + \right. \\ & \quad C_\sigma (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)}^{\frac{4}{3}} + \|\xi\|_{L_t^\infty(L^\infty)}^4 + \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} + \|\xi\|_{L_t^\infty(L^\infty)}^2 + \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} \\ & \quad \left. + \|u\|_{L_t^\infty(L^q)}^{\frac{2q}{q-2}} + \|\xi\|_{L_t^\infty(L^\infty)}^2 + \|z\|_{L_t^\infty(L^\infty)}) \right) \times (\|u\|_{H^{s_1}} + \|z\|_{H^{s_2}} + \|\xi\|_{H^{s_2+1}}) dt' \\ & \quad + (1+t) \sigma (\|z\|_{\tilde{L}_t^1(H^{s_2+2})} + \|\xi\|_{\tilde{L}_t^1(H^{s_2+3})}). \end{aligned}$$

Choose $\sigma = c(1+T^*)^{-1}$. Lemmas 3.1-3.7 and the inequality (3.34) imply that

$$C_\sigma (\|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)}^{\frac{4}{3}} + \|\xi\|_{L_t^\infty(L^\infty)}^4 + \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)} + \|\xi\|_{L_t^\infty(L^\infty)}^2 + \|\nabla \cdot \xi\|_{L_t^\infty(L^\infty)})$$

$$+ \|u\|_{L^\infty(L^q)}^{\frac{2q}{q-2}} + \|\xi\|_{L_t^\infty(L^\infty)}^2 + \|z\|_{L_t^\infty(L^\infty)} \leq C(T^*) < \infty,$$

from which it follows that

$$\begin{aligned} (3.49) \quad & \|u\|_{\tilde{L}_t^\infty(H^{s_1})} + \|z\|_{\tilde{L}_t^\infty(H^{s_2})} + \|\xi\|_{\tilde{L}_t^\infty(H^{s_2+1})} \\ & \leq C(T^*) \left((\|u_0\|_{H^{s_1}} + \|z_0\|_{H^{s_2}} + \|\xi_0\|_{H^{s_2+1}}) + \int_0^t (\|u\|_{C^{0,1}} + \|\nabla z\|_{L^\infty} + 1) \right. \\ & \quad \times (\|u\|_{\tilde{L}_{t'}^\infty(H^{s_1})} + \|z\|_{\tilde{L}_{t'}^\infty(H^{s_2})} + \|\xi\|_{\tilde{L}_{t'}^\infty(H^{s_2+1})}) dt' \Big) \\ & \triangleq B(t). \end{aligned}$$

Denote

$$\begin{aligned} & \|u_0\|_{H^{s_1}} + \|z_0\|_{H^{s_2}} + \|\xi_0\|_{H^{s_2+1}} \triangleq A_0, \\ & \|u\|_{\tilde{L}_T^\infty(H^{s_1})} + \|z\|_{\tilde{L}_T^\infty(H^{s_2})} + \|\xi\|_{\tilde{L}_T^\infty(H^{s_2+1})} \triangleq A(t), \end{aligned}$$

Let $\epsilon = \min(1, s_1 - 2)$ and $\Gamma(r) = 1 + \log r : [1, \infty) \rightarrow [0, \infty)$ be the function associated with the modulus of continuity $\mu(r) = r(1 - \log r)$. We can extend the domain of definition of Γ to $[0, \infty)$ with $\Gamma(s) = \Gamma(1) = 1$, for $0 \leq s < 1$. The function $G(y) \stackrel{\text{def}}{=} \int_1^y \frac{dy'}{\Gamma(y' \frac{1}{\epsilon}) y'} = \epsilon \log(1 + \frac{1}{\epsilon} \log y)$ then maps $[1, +\infty)$ onto and one-to-one $[0, +\infty)$.

Assuming that $A_0 > 0$, otherwise $(0, 0, 0)$ is the global solution. Using Lemma 2.22 with $\Lambda = A_0$, we get

$$\begin{aligned} B(t) & \leq C(T^*) \left(A_0 + \int_0^t (\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla z\|_{L^\infty} + 1) B(t') dt' \right) \\ & \leq C(T^*) \left(A_0 + \int_0^t \left\{ \|u\|_{B_{\infty,\infty}^1} + \|\nabla z\|_{L^\infty} + 1 + C_\epsilon \left(\|u\|_{C_\mu} + A_0 \right) \left(1 + \Gamma \left(\left(\frac{\|\nabla u\|_{C^{0,\epsilon}}}{\|u\|_{C_\mu} + A_0} \right)^{\frac{1}{\epsilon}} \right) \right) \right\} B(t') dt' \right) \\ & \leq C(\epsilon, T^*) \left(A_0 + \int_0^t \left\{ (\|u\|_{B_{\infty,\infty}^1} + \|\nabla z\|_{L^\infty} + 1 + A_0) \left(1 + \Gamma \left(\left(\frac{C\|u(t')\|_{H^{s_1}}}{A_0} \right)^{\frac{1}{\epsilon}} \right) \right) \right\} B(t') dt' \right) \\ & \leq C(\epsilon, T^*) \left(A_0 + \int_0^t \left(\|u\|_{B_{\infty,\infty}^1} + \|\nabla z\|_{L^\infty} + 1 + A_0 \right) \Gamma \left(\left(\frac{CB(t')}{A_0} \right)^{\frac{1}{\epsilon}} \right) B(t') dt' \right) \\ & \triangleq \frac{R(t)A_0}{C}, \end{aligned}$$

where we have used $B_{\infty,\infty}^1 \hookrightarrow L^\infty$, $B_{\infty,\infty}^1 \hookrightarrow C_\mu$, $H^{s_1} \hookrightarrow C^{0,\epsilon}$, $\|u(t')\|_{H^{s_1}} \leq B(t')$ and C has been chosen large enough such that $R(t) = \frac{B(t)C}{A_0} \geq C > 1$.

Because the function Γ is nondecreasing, after a few computations, we have that

$$\frac{d}{dt} R(t) \leq \Gamma(R(t)^{\frac{1}{\epsilon}}) R(t) C(\epsilon, T^*) (\|u(t)\|_{B_{\infty,\infty}^1} + \|\nabla z(t)\|_{L^\infty} + 1 + A_0),$$

thus

$$\frac{d}{dt} G(R(t)) \leq C(\epsilon, T^*) (\|u(t)\|_{B_{\infty,\infty}^1} + A_0 + 1 + \|\nabla z(t)\|_{L^\infty}).$$

Integrating then gives

$$R(t) \leq G^{-1} \left(G(R(0)) + \int_0^t C(\epsilon, T^*) (\|u\|_{B_{\infty,\infty}^1} + A_0 + 1 + \|\nabla z\|_{L^\infty}) dt' \right) < \infty,$$

where we have used Lemmas 3.6-3.7. Therefore, $\|u(t)\|_{H^{s_1}}$, $\|z(t)\|_{H^{s_2}}$, and $\|\xi(t)\|_{H^{s_2+1}}$ stay bounded on $[0, T^*)$. The local existence part of Theorem 1.1 then enables us to extend the solution beyond T^* , which stands in contradiction to the definition of T^* . Hence $T^* = +\infty$. This completes the proof of the theorem. \square

4 Proof of Theorem 1.3

To begin, we denote by BMO the space of functions of bounded mean oscillations. It is well known that BMO strictly includes L^∞ . We introduce the following Hardy-Littlewood-Sobolev inequality.

Lemma 4.1. [6] For $0 < \gamma < d$, the operator $(-\Delta)^{\frac{\gamma}{2}}$ is bounded from the Hardy space \mathcal{H}^1 to $L^{\frac{d}{d-\gamma}}$ and from $L^{\frac{d}{\gamma}}$ to BMO .

4.1. Global existence for the ENPP system

Let $(u_0, z_0, \xi_0) = (u_0, n_0 + p_0, -\nabla(\Delta)^{-1}(n_0 - p_0))$. According to Theorem 1.1, there exists a global solution (u, z, ξ) satisfies the system (03) in the spaces defined as in Theorem 1.1. Since $\mathcal{L}\xi = \xi$, it is then easy to that $(u, n, p) = (u, \frac{z+\nabla \cdot \xi}{2}, \frac{z-\nabla \cdot \xi}{2})$ solves the system (02).

Denote

$$\phi_0 \triangleq -(-\Delta)^{-1} \nabla \cdot \xi.$$

As $\xi \in L^\infty(\mathbb{R}^+; L^2)$, applying Lemma 4.1 with $d = 2$ and $\gamma = 1$ implies that $\phi_0 \in L^\infty(\mathbb{R}^+; BMO)$. Thanks again to the fact that $\mathcal{L}\xi = \xi$, we have $\nabla \phi_0 = \mathcal{L}\xi = \xi$, and $\Delta \phi_0 = \nabla \cdot \xi = n - p$. Similarly, let

$$P_0 \triangleq P_\pi(u, u) - (-\Delta)^{-1} \nabla \cdot ((\nabla \cdot \xi)\xi),$$

where $P_\pi(u, u) \in L^\infty(\mathbb{R}^+; H^{s_1+1})$ is defined as in Lemma 2.16. Note that $\xi \in L^\infty(\mathbb{R}^+; H^{s_1+1})$ with $s_1 > 1$ implies that $\nabla \cdot \xi \in L^\infty(\mathbb{R}^+; L^2)$ and $\xi \in L^\infty(\mathbb{R}^+; L^\infty)$. Again using lemma 4.1, we get

$$P_0 \in L^\infty(\mathbb{R}^+; H^{s_1+1} + BMO) \hookrightarrow L^\infty(\mathbb{R}^+; L^\infty + BMO) \hookrightarrow L^\infty(\mathbb{R}^+; BMO).$$

Finally, it is easy to see that $(u, \frac{z+\nabla \cdot \xi}{2}, \frac{z-\nabla \cdot \xi}{2}, P_0, \phi_0)$ satisfies the *ENPP* system.

4.2. Uniqueness for the ENPP system

Suppose that there exists a global solution (u, n, p, P, ϕ) satisfying the *ENPP* system in the spaces defined as in Theorem 1.3. We first show that

$$\nabla \Phi = -\nabla(-\Delta)^{-1}(n - p) \triangleq \xi, \text{ and } \nabla P = \pi(u, u) + (I - \mathcal{P})((n - p)\nabla(-\Delta)^{-1}(p - n)).$$

In fact, Let ϕ_0, P_0 be defined as in the above subsection. As $\Delta \phi = n - p = \Delta \phi_0$, hence $\phi - \phi_0$ is a harmonic polynomial. Note that $\phi \in L^\infty(\mathbb{R}^+; BMO)$ is required in Theorem 1.3 and $\phi_0 \in L^\infty(\mathbb{R}^+; BMO)$ is illustrated before. Thus $\phi - \phi_0$ depends only on t , and

$$(4.1) \quad \nabla \phi = \nabla \phi_0 = \xi = -\nabla(-\Delta)^{-1}(n - p).$$

Next applying the operator $\nabla \cdot$ to the first equation of the *ENPP* system, we get

$$-\Delta P = \nabla \cdot (u \cdot \nabla u) - \nabla \cdot ((\nabla \cdot \xi)\xi) = -\Delta P_0.$$

Note that $P - P_0$ is in $L^\infty(\mathbb{R}^+; BMO)$. Similar arguments as that for $\phi - \phi_0$ yield that

$$\nabla P = \nabla P_0 = \Pi(u, u) - \nabla(-\Delta)^{-1} \nabla \cdot ((\nabla \cdot \xi)\xi) = \Pi(u, u) + (I - \mathcal{P})((n - p)\nabla(-\Delta)^{-1}(p - n)).$$

Next it is easy to see that (u, n, p, ξ) solves the system (02), and $(u, n + p, \xi)$ solves the system (03). The uniqueness of the system (03) in Theorem 1.1 then implies that $(u, n, p, \nabla P, \nabla \phi)$ is uniquely determined by the initial data.

This completes the proof of the theorem. \square

Acknowledgements. This work was partially supported by NNSFC (No. 11271382), RFDP (No. 20120171110014), and the key project of Sun Yat-sen University.

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